The idea of partial fractions is to “undo the operation of combining fractions over a common denominator.” At first this seems like a pointless task, but in the context of integration it becomes an invaluable tool. For example,

\[
\frac{2x}{(x^2 + 1)^2} + \frac{3x^2 + 2x}{x^3 + x^2 + 5}
\]

is much easier to integrate than

\[
\frac{3x^6 + 2x^5 + 8x^4 + 6x^3 + 3x^2 + 12x}{x^7 + x^6 + 2x^5 + 7x^4 + x^3 + 11x^2 + 5},
\]

but they are in fact the same expression. In your studies you may have to integrate something like the second expression, so it’d be nice to have a method to move from it to the first expression which you can then integrate using a \(u\)-substitution on each term in the sum. This is what the method of partial fractions is all about.

Here are the steps that you should follow to successfully guess what the partial fraction decomposition should look like.

\textbf{Step 1.} Start by checking that the degree of the numerator is strictly less than the degree of the denominator. In most problems, this is already the case. If it’s not, you’ll have to use polynomial long division and do partial fractions on the remainder term instead. For example, the rational expression

\[
\frac{x^3 + 3x^2 - 2x + 1}{x^2 - x + 2}
\]

can be written as

\[
x + 4 - \frac{7}{x^2 - x + 2}.
\]

The first two terms are easy to integrate using the power rule, and the last term in the sum is what you would want to do partial fractions on.

\textbf{Step 2.} Factor out the denominator as far as possible. This includes factoring quadratics if possible. Recall from algebra that a good way to check whether a quadratic \(ax^2 + bx + c\) is factorable is to compute the discriminant \(b^2 - 4ac\) – if it’s a non-negative perfect square (e.g. \(9 = 3^2\) or \(\frac{4}{25} = (\frac{2}{5})^2\), you can factor it, and if it’s not then the quadratic is said to be \textit{irreducible}. Group all of your factors together.

\textbf{Step 3.} Next, look at each of the individual factors from step 2 and classify them as being in one of these cases (the first four are also in your textbook):
Case 1. (Distinct linear factor \( mx + b \)): In this case, the factor gets the term

\[ \frac{A}{mx + b} \]

Case 2. (Repeated linear factor, \((mx + b)^2\)): In this case, the factor gets two terms,

\[ \frac{A}{mx + b} + \frac{B}{(mx + b)^2} \]

Notice that the coefficients on top are both just constants. There are no \( x \) terms in either of them. Even though \((mx + b)^2\) is a quadratic polynomial when you square it out, the factor itself, \( mx + b \), is just linear.

Case 3. (Distinct quadratic factor, \( ax^2 + bx + c \)): The factor will only get one term,

\[ \frac{Ax + B}{ax^2 + bx + c} \]

Notice that this time, the numerator is a linear term (degree 1, which is one degree less than the degree of the denominator) and contains all powers of \( x \) which are less than or equal to 1. This is how the pattern will go in case 5 (the general case). It may turn out that the \( B \) is equal to zero, but we have to account for the fact that it may not be.

Case 4. (Repeated quadratic factor, \((ax^2 + bx + c)^2\)): As in case 2, the factor will get two terms,

\[ \frac{Ax + B}{ax^2 + bx + c} + \frac{Cx + D}{(ax^2 + bx + c)^2} \]

As in case 3, the numerators get degree 1 terms which include all powers of \( x \) which are less than or equal to 1. Also, as in case 2, even though \((ax^2 + bx + c)^2\) is a quartic (degree 4) term when you expand it out, the factor itself is a quadratic term which is why it only gets \( Cx + D \) in the numerator.

Case 5. General case: Suppose you have an \( n \)th degree polynomial with multiplicity \( m \) (this means that it’s a polynomial with an \( x^n \) term as its maximum power of \( x \), and the factor appears \( m \) times. Then you will get \( m \) terms in the decomposition, and each of them will have numerators which are \( n - 1 \) degree polynomials with unknown coefficients. For example, if we had the factor

\[(x^3 + 2)^2,\]
then the degree is \( n = 3 \) and its multiplicity is \( m = 2 \). Therefore we’d get two terms, each with a degree \( n - 1 = 2 \) term in the numerator. Therefore the partial fraction decomposition for this term is

\[
\frac{Ax^2 + Bx + C}{x^3 + 2} + \frac{Dx^2 + Ex + F}{(x^3 + 2)^2}
\]

As a more complicated example, consider

\[(x^4 + 3x^3 - 2x + 4)^5.\]

The degree of the polynomial is \( n = 4 \), and its multiplicity is \( m = 5 \). Thus the partial fraction decomposition has five terms, and each of them will have a degree \( n - 1 = 3 \) polynomial in the numerator. Its decomposition is

\[
\frac{Ax^3 + Bx^2 +Cx + D}{x^4 + 3x^2 - 2x + 4} + \frac{Ex^3 + Fx^2 + Gx + H}{(x^4 + 3x^2 - 2x + 4)^2} + \frac{Ix^3 + Jx^2 + Kx + L}{(x^4 + 3x^2 - 2x + 4)^3} + \frac{Mx^3 + Nx^2 + Ox + P}{(x^4 + 3x^2 - 2x + 4)^4} + \frac{Qx^3 + Rx^2 + Sx + T}{(x^4 + 3x^2 - 2x + 4)^5}
\]

There are a lot of unknown coefficients here (20 of them!) so thankfully things this complicated hardly ever appear (and if they do, you can use a computer to find the coefficients for you).

In many of the problems that you’ll have, you need to mix and match the cases. For the expressions

\[
\frac{x^2 + 4x - 7}{(x + 1)(x^2 + 4)} \quad \text{and} \quad \frac{x^3 - 5x + 6}{(x^2 + 2x + 2)^2(x + 1)(x - 2)^2(x^2 + 1)}
\]

the first is a mixture of cases 1 and 3, whereas the second one is a mixture of the first four cases. For the latter expression, one would want to guess

\[
\begin{align*}
\text{Case 4} & : \frac{Ax + B}{x^2 + 2x + 2} + \frac{Cx + D}{(x^2 + 2x + 2)^2} \\
\text{Case 1} & : \frac{E}{x + 1} \\
\text{Case 2} & : \frac{F}{x - 2} + \frac{G}{(x - 2)^2} \\
\text{Case 3} & : \frac{Hx + I}{x^2 + 1}
\end{align*}
\]

**Step 4.** Finally, combine your guess over the least common denominator and set it equal to the original expression. This amounts to having equality of the numerators. We discussed a few different ways that one can go about finding the coefficients at this point. The surefire way is to expand everything out each side, group by powers of \( x \), and compare the coefficients. But sometimes this is really tedious, and in the case of linear factors, one can sometimes simplify the process by plugging in “nice” values of \( x \).
which will cancel out a lot of terms (for example, if there’s an $x - 2$ factor, then plugging in $x = 2$ will kill all of the coefficients which have that as a factor and leave you with a smaller number of unknowns to deal with).

Don’t forget to integrate the result (if requested in the problem) at the very end! Usually (but not always), this just involves a $u$-sub, with $u$ being the denominator, and $du$ a multiple of the numerator.

**Exercises.** Find the partial fraction decompositions of the following functions. Solve for the coefficients unless otherwise indicated, but don’t integrate the result. Answers are posted at [http://www.math.purdue.edu/~krotz/teaching](http://www.math.purdue.edu/~krotz/teaching).

(a) \( \frac{51 - 7x}{x^2 - 3x - 18} \)

(b) \( \frac{-3x^2 - 4x - 7}{(x^2 + 2x + 1)(x - 2)} \)

(c) \( \frac{3x^2 + 5x + 4}{(x^2 + 2x + 2)(x + 1)} \)

(d) \( \frac{-9x^6 + x^5 + 8x^4 - 34x^3 + 15x^2 + 9x - 54}{(x^3 + 3)^2(x + 3)^2} \) (Just write the decomposition.)