Theorem (*known as the Principle of Superposition*): Consider the second-order, linear, homogeneous ordinary differential equation

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0. \qquad (*)$$

If y_1 and y_2 are both solutions to $(*)$, then for any two constants c_1 and c_2,

$$y = c_1y_1 + c_2y_2$$

is also a solution to $(*)$.

Proof: The fact that y_1 and y_2 are solutions to $(*)$ imply that

$$p(x)y_1''(x) + q(x)y_1'(x) + r(x)y_1(x) = 0 \qquad \text{and} \tag{1}$$
$$p(x)y_2''(x) + q(x)y_2'(x) + r(x)y_2(x) = 0. \tag{2}$$

Since c_1 and c_2 are constants, we have

$$\begin{aligned} y'(x) &= c_1y_1'(x) + c_2y_2'(x), \qquad \text{and} \\ y''(x) &= c_1y_1''(x) + c_2y_2'(x). \end{aligned}$$

Inserting these into $(*)$, we see that

$$\begin{aligned} p(x)y''(x) + q(x)y'(x) + r(x)y(x) &= p(x)\big(\overbrace{c_1y_1''(x) + c_2y_2'(x)}^{y''(x)}\big) + q(x)\big(\overbrace{c_1y_1'(x) + c_2y_2'(x)}^{y'(x)}\big) \\ &\quad + r(x)\big(\underbrace{c_1y_1(x) + c_2y_2(x)}_{y(x)}\big) \end{aligned} \tag{3}$$

We now regroup the terms in (3) by those terms with c_1's and c_2's:

$$\begin{aligned} p(x)y''(x) + q(x)y'(x) + r(x)y(x) &= c_1p(x)y_1''(x) + c_2p(x)y_2'(x) + c_1q(x)y_1'(x) + c_2q(x)y_2'(x) \\ &\quad + c_1r(x)y_1(x) + c_2r(x)y_2(x) \\ &= c_1\left[p(x)y_1''(x) + q(x)y_1'(x) + r(x)y_1(x)\right] \\ &\quad + c_2\left[p(x)y_2''(x) + q(x)y_2'(x) + r(x)y_2(x)\right] \end{aligned} \tag{4}$$

By equations (1) and (2), the right-hand side of (4) is zero. In other words,

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0,$$

so that y is a solution to $(*)$.