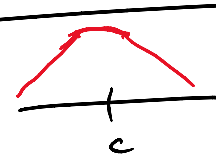
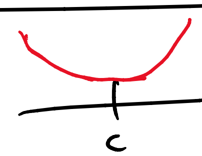
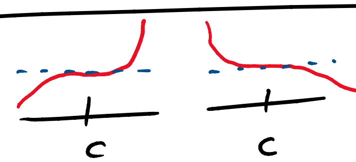
☆ What Derivatives Tell Us (Part 2)

Warm Up → Announcements

Last class

First Derivative Test:

At $x=c$ where $f'(c)=0$ or $f'(c)=\text{DNE}$ ← c is a critical point

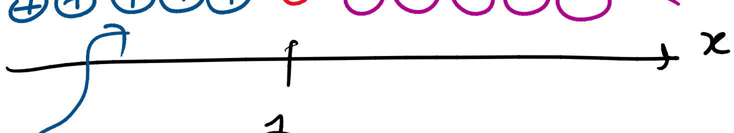
Then	$f'(x)$	plot $f(x)$	c
(i)	$f'(x)$ changes from $\oplus$ to $\ominus$ across c		c is a local max
(ii)	$f'(x)$ changes from $\ominus$ to $\oplus$ across c		c is a local min
(iii)	$f'(x)$ doesn't change sign across c		c is neither a local max nor a local min

Ex: Apply the 1st Deriv Test to $f(x)=xe^{-x}$

warm-up → critical point $c=1$

Draw a number line + chart the sign $f'(x)=(1-x)e^{-x}$

$f'(x)$ $\oplus\oplus\oplus\oplus\oplus$ 0 $\ominus\ominus\ominus\ominus\ominus$ ← Hot Seat Poll 1



sign of $f'(x)$

Check $x=0$

$\boxed{\text{1st Deriv Test: at } c=1 \text{ is local max}}$

Check $x = 0$

$f'(0) = (1-0)e^{-0} = 1 \cdot 1$ ⊕

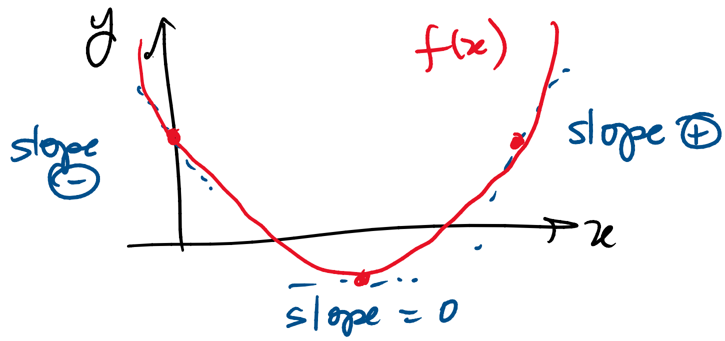
$f'(-1) = (1-(-1))e^{-1} = \frac{2}{e}$ ⊕

at $c=1$ is local max

Q: What does $f''(x)$ tell us?

A: If $f'' > 0 \rightarrow (f')' > 0 \Rightarrow f'$ is increasing

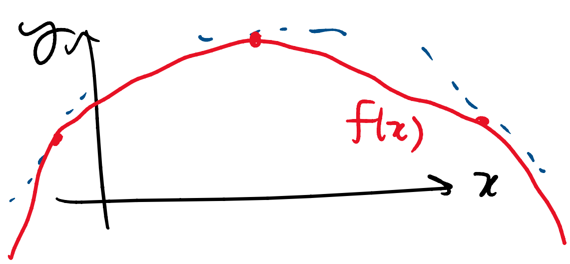
slope of the tangent line to $f(x)$ is increasing



When $f'' > 0$ we say that the graph of f is concave up

If $f'' < 0 \rightarrow (f')' < 0 \Rightarrow f'$ is decreasing

slope of tangent line is decreasing

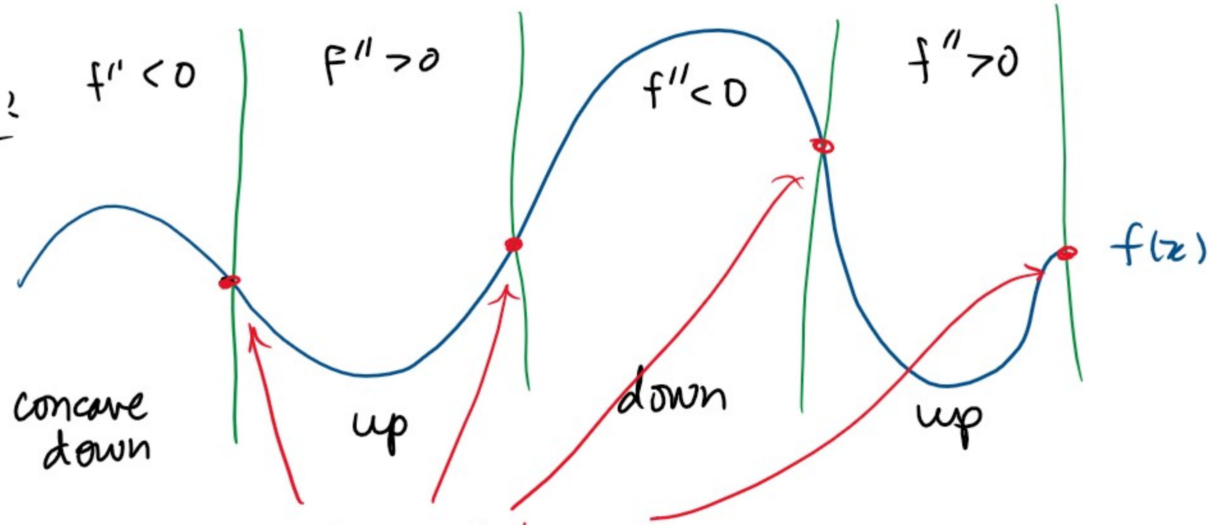


When $f'' < 0$ we say the graph is concave down

Def: The point where the concavity changes is called an inflection point

Def: ... called an inflection point

Ex:



inflection points

when $f''(x)=0$

Ex: On what intervals is $f(x) = xe^{-x}$ concave up / down? Where are the inflection points

$$f(x) = xe^{-x}$$

$$f'(x) = (1-x)e^{-x}$$

Product Rule →

$$f''(x) = (-1)e^{-x} + (1-x)(-e^{-x})$$

$$= (-1-(1-x))e^{-x}$$

$$= (-1-1+x)e^{-x}$$

$$f''(x) = (-2+x)e^{-x}$$

Find the inflection points $f''(x)=0$

$$(-2+x)e^{-x} = 0$$

$$-2+x=0 \qquad \cancel{e^{-x}=0}$$

$$\boxed{x=2}$$

$$\boxed{\text{inflection point at } x=2}$$

Chart the sign of $f''(x)$

f'' $-\ -\ -\ -\ -\quad 0 \quad +\ +\ +\ +\ +$ $\rightarrow x$

$x=2$

Check at $x=0$

$f''(0) = (-2+0)e^{-0} = -2$ $-$

check $x=3$

$f''(3) = (-2+3)e^{-3} = \frac{1}{e^3}$ $+$

$- \ -$ ← $f''<0$ concave down

$+ \ +$ ← concave up

$$\boxed{\begin{array}{l} \text{so } f(x) \text{ is concave up on } (2,\infty) \\ \text{concave down on } (-\infty,2) \\ \text{inflection point at } x=2 \end{array}}$$

NOTE: We can also use $f''(x)$ to find local min or max

critical point of $f(x) = xe^{-x}$ → $c=1$

Check f'' @ the critical point

$$f''(1) = (-2+x)e^{-x}\Big|_{x=1} = (-2+1)e^{-1} = \frac{-1}{e} \quad -$$

$- \ -$ ← concave down

⇒ local max

Second Derivative Test

Given $f(x)$ and $f'(c) = 0$

Then:

$f''(c)$	c	plot
$f''(c) > 0$ (+)	$x=c$ is a local min	$+\ +$
$f''(c) < 0$	$x=c$ is a local max	$-\ -$
$f''(c) = 0$	the test is inconclusive not enough info	

Comparing 1st Derivative Test & 2nd Derivative Test

- 1st deriv test always works
  even if $f'(c)$ is undefined
  $\underbrace{\hphantom{xxxxxxxxxxxx}}$ 2nd deriv test, need $f'(c) = 0$

- 2nd deriv test is sometimes faster
  don't need to chart f'
  - good when we know f' is diff'ble
    - polynomials
    - $\sin(x)$, $\cos(x)$, e^x

Ex: $f(x) = x - 3x^{2/3}$ test to find
use the 2nd Deriv local max/min

Ex: $f(x) = x - \ldots$

use the 2nd Deriv test to find the local max/min

critical points: $f'(x) = 0$ or $f'(x) = DNE$

$$f'(x) = 1 - \cancel{3}\left(\frac{2}{\cancel{3}}\right)x^{-1/3} = 1 - \frac{2}{x^{1/3}}$$

undefined at $\boxed{x = 0}$

$$1 - \frac{2}{x^{1/3}} = 0$$

$$1 = \frac{2}{x^{1/3}}$$

$$\left(x^{1/3}\right)^3 = (2)^3$$

$$\boxed{x = 8}$$

$$\boxed{\text{critical points } x = 0, 8}$$

2nd Deriv: $f''(x) = 0 - 2\left(-\frac{1}{3}\right)x^{-4/3}$

$$= \frac{2}{3}x^{-4/3} = \frac{2}{3x^{4/3}}$$

@ $x = 8$ $\quad f''(8) = \frac{2}{3}(8)^{-4/3} = \frac{2}{3}\left(2^{\cancel{3}}\right)^{-4/\cancel{3}} = \frac{2}{3} \cdot 2^{-4}$

$$= \frac{1}{3 \cdot 2^3} = \frac{1}{24} \; +$$

f'' + at $x = 8$

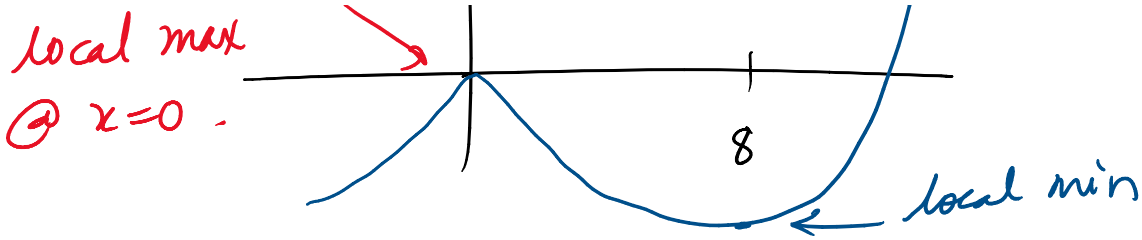
++ $\rightarrow$ $\boxed{x = 8 \text{ is a local min}}$

@ $x = 0$ $\quad f''(0) = \frac{2}{3}(0)^{-4/3} = \frac{2}{3 \cdot 0^{4/3}}$ undefined

2nd Deriv test does not apply

1st deriv test

local max


local max
@ x=0.
8
local min