★ Optimization – Part 1

Warm Up → Announcements

Optimization:

GOAL: use calculus to find the maximum or minimum of a quantity.

Ex: Let $x+y=10$ where $x, y \geq 0$
Find x and y such that the product $P = xy$ is maximized.

objective: quantity to be max or min

GOAL: max $P = xy$

CONDITION: $x+y=10$ ← constraint: condition that the variables must meet.

DOMAIN: $x, y \geq 0$

Obj: Max $P = xy$ ← normally, find P' and critical points, 1st/2nd Deriv Test (need $f(x)$ a function of 1 variable)

function of 2 variables

Use the constraint to eliminate one variable

$$x + y = 10$$
$$y = 10 - x$$

plug into the objective:

Obj: max $P = xy = x(10-x)$

Domain: $x \geq 0$

$$y \geq 0$$
$$10 - x \geq 0$$
$$10 \geq x$$

$$0 \leq x \leq 10$$

critical point is $x=5$

max $P(x)$ over $[0,10]$

Evaluate $P(x)$ at endpoints + critical point

$P(0) = 0 \cdot (10-0) = 0$

$P(5) = 5 \cdot (10-5) = 25$

$P(10) = 10 \cdot (10-10) = 0$

Maximum product when
$x=5$
$y=10-5=5$

Optimization Strategy:

① Draw a picture + label variables

② Identity

Objective $P = xy$

Constraint $x+y=10$

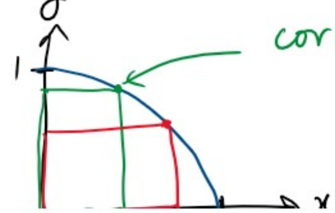
Domain $x, y \geq 0$

③ Solve the constraint for one vaiable + plug into the objective $P = x(10-x)$

④ use calculus tools to find max/min of equation in step ③

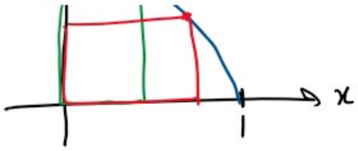
⑤ check your answer

Ex: What is the area of the largest rectangle that can be inscribed inside the 1st quadrant portion of the unit circle?

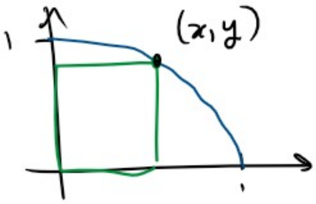


corner stays on the circle

where should the corner be to maximize the area of the rectangle?



to maximize the area of the rectangle?



Obj: max $A = xy$

Constraint $x^2+y^2=1$

Domain: $0 \le x \le 1$
$0 \le y \le 1$

(Given $x,y \ge 0$ b/c 1st quadrant)

Eliminate on variable:

$$x^2+y^2=1$$
$$y=\sqrt{1-x^2}$$

Plug into Obj: max $A(x) = x\sqrt{1-x^2}$

Find critical points: $A'(x)=0$ and $A'(x)$ DNE

$$A(x) = \sqrt{x^2(1-x^2)} = \sqrt{x^2-x^4}$$

$$A'(x) = \frac{1}{2}(x^2-x^4)^{-1/2}(2x-4x^3)$$

$$= \frac{\cancel{2}x - \cancel{4}^{2}x^3}{\cancel{2}\sqrt{x^2-x^4}} = \frac{\cancel{x}-2x^{\cancel{3}\,2}}{\cancel{x}\sqrt{1-x^2}}$$

$A'(x)$ DNE
@ $x=1$
denom $=0$

$$A'(x) = \frac{1-2x^2}{\sqrt{1-x^2}} = 0$$

$$1-2x^2=0$$
$$\frac{1}{2}=x^2 \qquad \boxed{x=\frac{1}{\sqrt{2}}}$$

Two critical points: $x=\frac{1}{\sqrt{2}}$, $x=1$ on $[0,1]$
$0 \le x \le 1$

1st Deriv Test: chart $f'(x)$

Evaluate $A(x)$ at endpoints + critical points

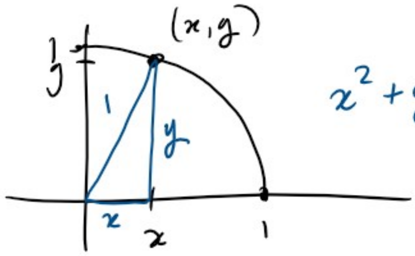
$$A(0) = 0\sqrt{1-0^2} = 0$$

$$A\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{1}{2}} = \frac{1}{2}$$

$$A\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{1}{2}} = \frac{1}{2}$$

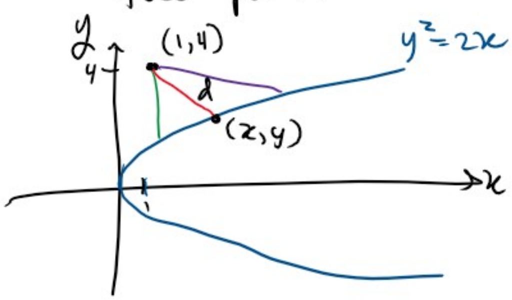
$$A(1) = 1\cdot\sqrt{1-1^2} = 0$$

max area when $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$



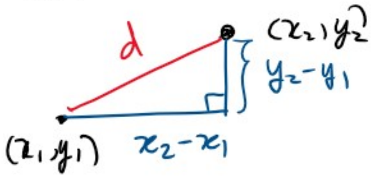
$x^2 + y^2 = 1^2$

Ex: What is the minimum distance from the point $(1,4)$ to the curve $y^2 = 2x$?



Obj: min d

distance between (x_1,y_1) and (x_2,y_2)



$$(x_2-x_1)^2 + (y_2-y_1)^2 = d^2$$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

choose $(x_1,y_1) = (1,4)$
$(x_2,y_2) = (x,y)$

$$\begin{cases} \text{Obj.} & \min\ d = \sqrt{(x-1)^2+(y-4)^2} \\ \text{Constraint:} & y^2 = 2x \\ \text{Domain:} & x \geq 0,\quad -\infty \leq y \leq \infty \end{cases}$$

NOTE: d and d^2 will both have a minimum at the same point (x_*, y_*)

Equivalently we can min $d^2 = (x-1)^2 + (y-4)^2$

Eliminate one variable: $y = \sqrt{2x}$ or $x = \frac{y^2}{2}$

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only the pos half of the curve

easier derivative gives entire curve

$$\left\{ \begin{array}{l} \min \quad d^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 \\ \text{domain: } -\infty \le y \le \infty \end{array} \right.$$

$\rightarrow$ Need to use 1st or 2nd Deriv Test

Next steps:
- find the critical points
- use 1st/2nd deriv test
- calculate d