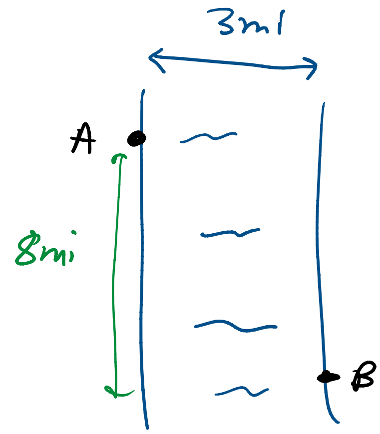
☆ Optimization – Part 2

Warm up → Announcements



Ex: Crossing a River

- River 3 mi wide
- Destination is 8mi downstream
- land speed is 8 mi/hr
- water speed is 6 mi/hr

Q: What is the fastest path from A to B?

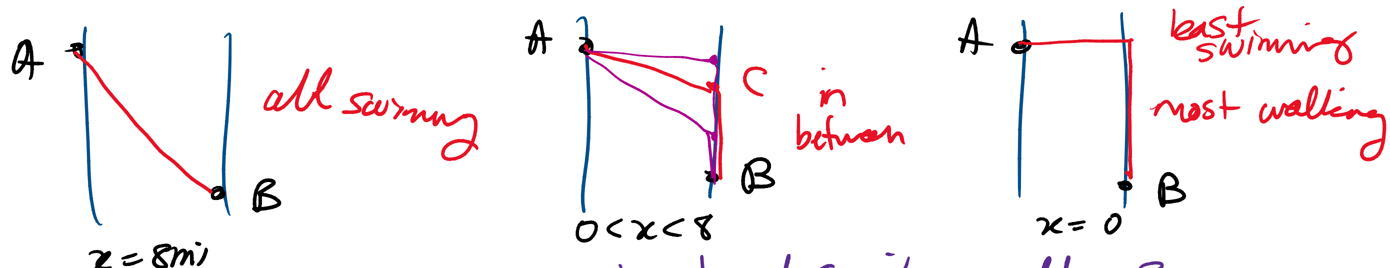
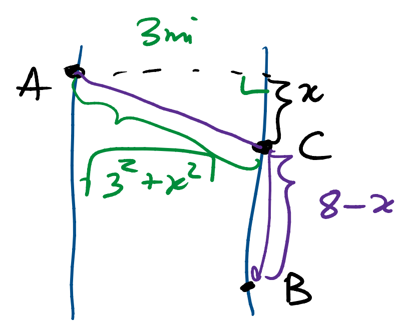


Figure out equations that describe all 3 scenarios



Distance: $\sqrt{9+x^2} + (8-x)$

Time: $\dfrac{\text{Distance}}{\text{Speed}}$

$$T(x) = \frac{\sqrt{9+x^2}}{6 \text{ mi/hr}} + \frac{(8-x)}{8 \text{ mi/hr}}$$

Objective: $\min \; T(x) = \dfrac{\sqrt{9+x^2}}{6} + \left(1 - \dfrac{x}{8}\right)$

Domain: $\quad 0 \le x \le 8$

Find the critical points:

$$T'(x) = 0$$

$$\frac{1}{6}\cancel{\frac{1}{2}}\left(9+x^2\right)^{-1/2}\left(\cancel{2}x\right) + \left(\frac{-1}{8}\right) = 0$$

$$\frac{x}{6\sqrt{9+x^2}} = \frac{1}{8}$$

$$\left(\frac{4x}{3}\right)^2 = 8\frac{x}{6} = \left(\sqrt{9+x^2}\right)^2$$

$$\left(\frac{16x^2}{9} = 9 + x^2\right)9$$

$$16x^2 = 81 + 9x^2$$

$$7x^2 = (16-9)x^2 = 81$$

$$x = \sqrt{x^2} = \sqrt{\frac{81}{7}} = \frac{\pm 9}{\sqrt{7}}$$

$$\boxed{x = \frac{9}{\sqrt{7}}}$$ critical point

Check $T(x)$ at the end points & critical point

$$T(0) = 1.5$$

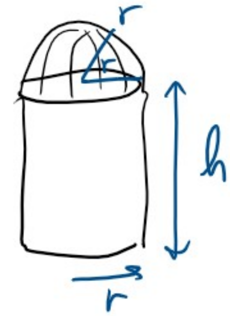
$$T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33$$

absolute min at $x = \frac{9}{\sqrt{7}}$

$$T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Fastest Path
swim $\frac{9}{\sqrt{7}}$ mi downstream
and walk rest

Ex: Building a Silo



- silo with semispherical roof
- needs to hold 10,000 ft^3
- sides cost \$2/ft^2 to build
- roof cost \$5/ft^2

Q: What shape should the silo be to minimize cost

Hint: lateral surface area of cylinder is: $2\pi r h$
surface area of hemisphere is $2\pi r^2$

Objective: min $C = 4\pi r h + 10\pi r^2$

Constraint: $V = 10{,}000\ ft^3$

$$\underbrace{\pi r^2 h}_{\text{cylinder Vol}} + \underbrace{\tfrac{2}{3}\pi r^3}_{\text{hemisphere Vol}} = 10{,}000$$

Domain: $r > 0 \quad h > 0$

Eliminate one variable using the constraint

$$\pi r^2 h + \tfrac{2}{3}\pi r^3 = 10{,}000$$

$$\pi r^2 h = 10000 - \tfrac{2}{3}\pi r^3$$

$$h = \frac{10000}{\pi r^2} - \frac{2\cancel{\pi} r^{\cancel{3}}}{3\cancel{\pi}\cancel{r^2}}$$

$$h = \frac{10000}{\pi r^2} - \frac{2}{3}r$$

plug into the objective

$$\min \quad C = 4\pi r h + 10\pi r^2$$

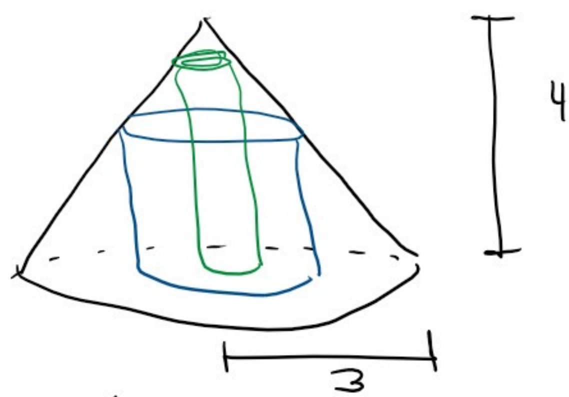
$$= 4\pi r\left(\frac{10000}{\pi r^2} - \frac{2}{3}r\right) + 10\pi r^2$$

$$= \frac{40000\cancel{\pi}\cancel{r}}{\cancel{\pi} r^{\cancel{2}}} - \frac{8\pi r^2}{3} + 10\pi r^2\left(\frac{3}{3}\right)$$

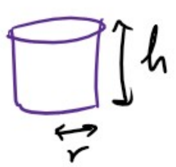
$$\boxed{\min \quad C = \frac{40000}{r} + \frac{22\pi r^2}{3} \quad \text{for } r > 0}$$

Next Steps:
- Find critical points $C'(r) = 0$
- find the absolute min
- use 1st / 2nd Deriv Test

Ex: Find the volume of the largest possible right cylinder that can be inscribed in a cone of height 4 and radius 3



$$\max \quad V = \pi r^2 h$$



Look at a cross-section

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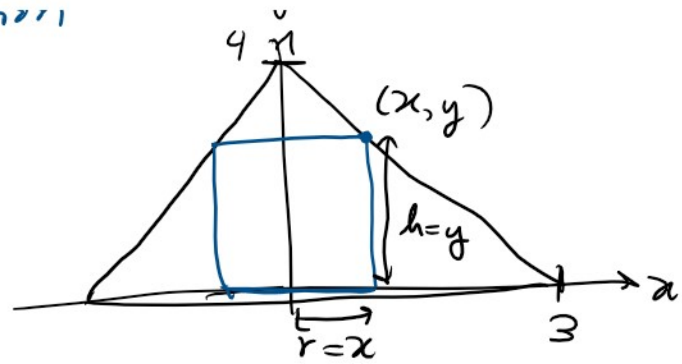
max $V = \pi x^2 y$



Objective: max $V = \pi x^2 y$

Constraint: $y = -\frac{4}{3}x + 4$

Domain: $0 \le x \le 3$ and $0 \le y \le 4$

Eliminate y by plugging in $y = -\frac{4}{3}x + 4$ into objective

$$\max \quad V = \pi x^2 \left(-\frac{4}{3}x + 4\right)$$

$$\max \quad V = -\frac{4}{3}\pi x^3 + 4\pi x^2$$

over $0 \le x \le 3$

Next Steps: - Find the critical points
- Find absolute maximum