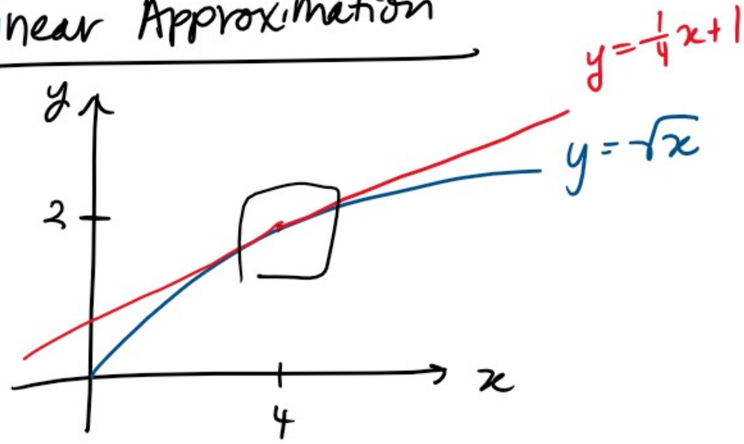
☆ Linear Approximation & Differentials

Warm Up → Announcements

Linear Approximation



very near (4,2) the tangent line and the curve f(x) are very close to each other

GOAL: Approximate $\sqrt{3.98}$

Idea: Use the tangent line $y=\frac{1}{4}x+1$

$$\sqrt{3.98} = f(3.98) \approx \left(\tfrac{1}{4}x+1\right)\Big|_{x=3.98}$$

$$= \tfrac{1}{4}(3.98)+1$$

$$= \frac{(4-0.02)}{4}+1$$

$$= 1-\left(\tfrac{0.02}{4}\right)+1$$

$$= 2 - 0.005$$

$$= \boxed{1.995}$$

True value

$\sqrt{3.98} =$
1.9949937

percent error: $\dfrac{|\text{approx} - \text{exact}|}{|\text{exact}|} \times 100$

$$= \frac{|1.995 - \sqrt{3.98}|}{} \times 100$$

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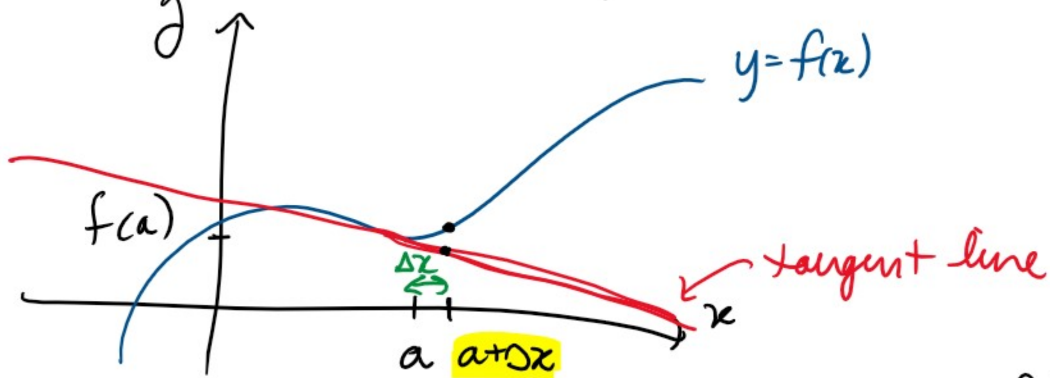
$$= 0.003$$

$\boxed{\text{percent error is } 0.003\%}$

Linear Approximation:

works for any curve $f(x)$
near a point $(a, f(a))$
→ f needs to be differentiable at $x = a$

use the tangent line to approximate $f(a + \Delta x)$



$x = a + \Delta x$

$$f(a+\Delta x) \approx f(a) + f'(a)(a+\Delta x - a)$$

$$= \underbrace{f(a) + f'(a)(x-a)}_{\text{equation of tangent line}}$$

$$= L(x)$$

↑ linear approximation

Linear Approx: $L(x) = f(a) + f'(a)(x-a)$

$L(x) \approx f(x)$ when x is close to a.

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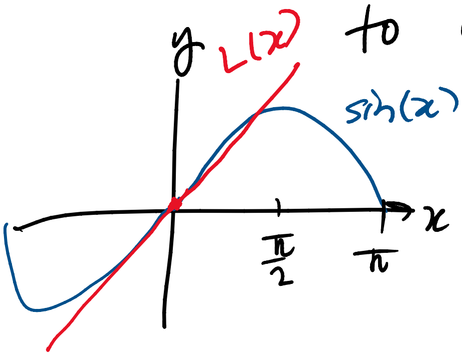
percent error: $\dfrac{|L(x) - f(x)|}{|f(x)|} \times 100$

Ex: Find a linear approximation of $f(x) = \sin(x)$ near $x = 0$ to estimate $\sin(0.1)$



$$L(x) = f(0) + f'(0)(x-0)$$

$$f(0) = \sin(0) = 0$$

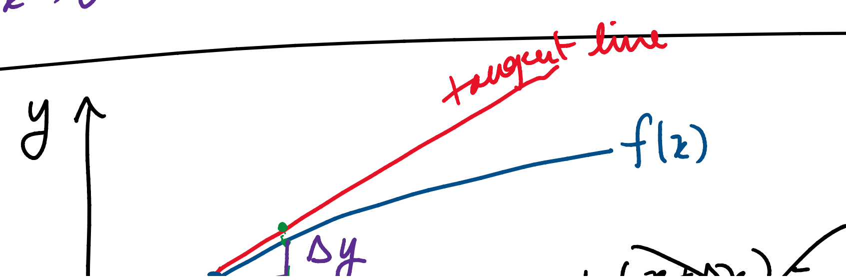
$$f'(0) = \big[\cos(x)\big]\Big|_{x=0} = 1$$

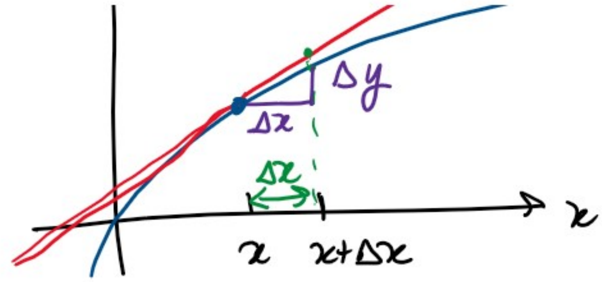
$$L(x) = 0 + 1(x-0) = x$$

Small Angle Approximation: $\sin(x) \approx x$ when x is small

NOTE: we can use this to show $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

Differentials:





$\cancel{L(x+\Delta x) = f(x) + f'(x)(x}$

$\Delta y = f(x+\Delta x) - f(x)$

$\Delta x = x+\Delta x - x$

$$\begin{aligned}
\Delta y &= f(x+\Delta x) - f(x) \\
&\approx L(x+\Delta x) - L(x) \\
&= \left[f(x) + f'(x)(\cancel{x}+\Delta x - \cancel{x})\right] \\
&\quad - \left[f(x) + f'(x)\cancel{(x-x)}\right] \\
&= \cancel{f(x)} + f'(x)\Delta x - \cancel{f(x)}
\end{aligned}$$

$$\boxed{\Delta y \approx f'(x)\Delta x}$$

Another way to think of it: $y = f(x)$

$$\Delta x \left(f'(x) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \right) \Delta x$$

<u>Ex</u>: A circle has radius 4, how much will the area of the circle change if the radius increases by 0.1?

Given: $r = 4$, $\Delta r = 0.1$ $A(r) = \pi r^2$

Want: $\Delta A \approx A'(4)\Delta r$

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Ex: Relativity:

The mass of an object changes as the velocity increases

m_0 - mass of object at rest

v - velocity of object

c - speed of light

mass of moving object $\dfrac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

Q: Can we find a linear approx when $v << c$?

"v is much less than c"

$\frac{v}{c} < 0.1$

Rename $x = \frac{v^2}{c^2}$ $\longrightarrow$ if $v << c$, x is close to zero

$$\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1-x}} = m_0 f(x)$$

$$f(x) = \frac{1}{\sqrt{1-x}} \approx L(x) = f(0) + f'(0)(x-0)$$

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$$f(0) = \frac{1}{\sqrt{1-0}} = 1$$

$$f'(0) = -\frac{1}{2}(1-x)^{-3/2}(-1)\Big|_{x=0}$$

$$= \frac{1}{2}(1-0)^{-3/2} = \frac{1}{2}$$

$$L(x) = 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x = 1 + \frac{x}{2}$$

$$\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \approx m_0 L(x) = m_0\left(1 + \frac{x}{2}\right)$$

$$= m_0\left(1 + \frac{v^2}{2c^2}\right)$$