☆ L'Hôpital's Rule:

Warm Up → Announcements

L'Hôpital's Rule:

GOAL: Evaluate limits when $\frac{0}{0}$ or $\frac{\infty}{\infty}$

TOOL: Derivative

Ex: $\lim_{x\to\infty} \frac{\ln(x)}{x-1} = \frac{\infty}{\infty}$ Indeterminate. Need to do more

But what?
- Factor?
- conjugate?
- algebra?

L'Hôpitals Rule:

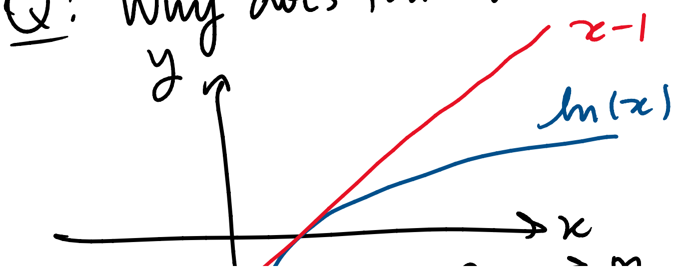
Idea: Take the derivative of top & bottom

$$\lim_{x\to\infty} \frac{\ln(x)}{x-1} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(x-1)} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0$$

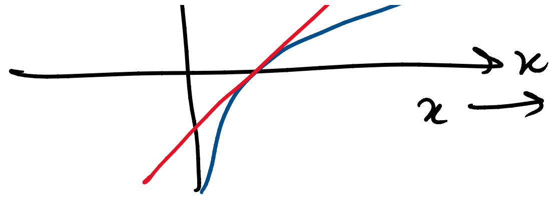
$$\boxed{\lim_{x\to\infty} \frac{\ln(x)}{x-1} = 0}$$

Q: Why does this work?



$\lim_{x\to\infty} \frac{\ln(x) \to \infty}{x-1 \to \infty}$

x-1 grows faster



$x \to \infty$

$x-1$ grows faster than $\ln(x)$

ratio of $\ln(x)/(x-1)$ goes to zero

use the derivative to compare the rates of change

L'Hôpital's Rule:

if $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} \to \frac{0}{0}$ or $\frac{\infty}{\infty}$

then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Other Indeterminate Forms:

$\infty - \infty$, $\infty \cdot 0$, 1^{∞}

Ex: $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) \to \frac{1}{0} - \frac{1}{1-1} = \infty - \infty \neq 0$

Indeterminate

Want to transform this into $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Find the common denominator

$$= \lim_{x\to 0^+} \left(\frac{1}{x}\frac{(e^x-1)}{(e^x-1)} - \frac{1}{e^x-1}\frac{(x)}{(x)} \right)$$

$$= \lim_{x\to 0^+} \left(\frac{e^x - 1 - x}{x(e^x-1)} \right) \to \frac{e^0 - 1 - 0}{0(e^0-1)} = \frac{0}{0}$$

$$\stackrel{L}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}(e^x - 1 - x)}{\frac{d}{dx}(x(e^x-1))} = \lim_{x\to 0^+} \frac{e^x - 1}{1\cancel{(e^x) + x(e^x +}}$$

$$\overset{L}{=} \lim_{x\to0^+} \frac{\frac{d}{dx}(e^x-1-x)}{\frac{d}{dx}(x(e^x-1))} = \lim_{x\to0^+} \frac{\quad}{\cancel{1(e^x)+x(e^x-1)}}$$

$$\rightarrow \frac{e^0-1}{1\cdot e^0+0(e^0-1)}$$

$$= \lim_{x\to0^+} \frac{e^x-1}{1(e^x-1)+x(e^x)}$$

$$= \frac{e^0-1}{1(e^0-1)+0(e^0)} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to0^+} \frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(e^x-1+xe^x)} = \lim_{x\to0^+} \frac{e^x}{e^x+1\cdot e^x+xe^x}$$

$$= \frac{e^0}{e^0+e^0+0} = \frac{1}{1+1} = \frac{1}{2}$$

$$\boxed{\lim_{x\to0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \frac{1}{2}}$$

Ex: $\lim_{x\to0^+} x\ln(x) \rightarrow 0\cdot(-\infty)$ Indeterminate

Want to rearrange so $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Rewrite $x = \frac{1}{\left(\frac{1}{x}\right)}$

$$= \lim_{x\to0^+} \frac{\ln(x)}{\left(\frac{1}{x}\right)} \rightarrow \frac{-\infty}{\infty} \quad \text{L'Hôpital's}$$

$$\overset{L}{=} \lim_{x\to0^+} \frac{\frac{1}{\cancel{x}}}{\left(\frac{-1}{x^{\cancel{2}}}\right)} = \lim_{x\to0^+}(-x) = \boxed{0}$$

Q? $\lim_{x\to0^+} 1\cdot\ln(x) + x\left(\frac{1}{x}\right) = -\infty$

$x \to 0^-$

Ex: $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \longrightarrow 1^{\infty}$ Indeterminate

Want to rewrite this in a way $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to apply L'Hôpitals Rule

But $f(x)^{g(x)} \longrightarrow$ use $\ln(\ )$

$$e^{\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)} = e^{\ln(L)} = L$$

$$L = e^{\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)}$$

$$= e^{\lim_{x\to\infty}\ln\left[\left(1+\frac{1}{x}\right)^{x}\right]}$$

$$= e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} \longrightarrow e^{\infty\cdot 0}$$ Indeterminate

Want $e^{\frac{0}{0}}$ or $e^{\frac{\infty}{\infty}}$

$x = \frac{1}{\left(\frac{1}{x}\right)}$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}} \to \frac{0}{0}$$

$$L = e^{\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(\cancel{\frac{-1}{x^2}}\right)}{\left(\cancel{\frac{-1}{x^2}}\right)}} = e^{\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}} = e^{1}$$

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e}$$