☆ Approximating Areas Under Curves

Warm Up → Announcements

Chapter 5: Integrals

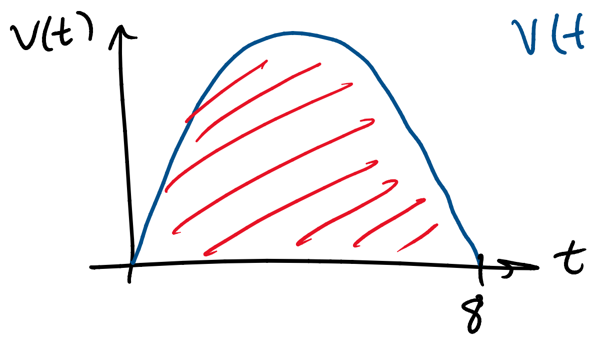
GOAL: Show the integral of a function is related to the area under its graph

KEY: Fundamental Theorem of Calculus

(will take several lectures)

Motivation: Suppose you are riding in a car
- can't see out window
- can only see the speedometer

Q: Can you determine how far you travelled?



$v(t) = t(8-t)$

Claim: the area between the curve and the t-axis is equal to the distance travelled

units: $[t] \times [v(t)]$

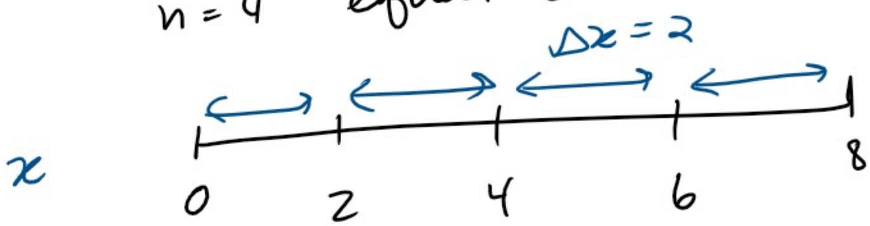
$s \times \frac{m}{s} = m$ ✓

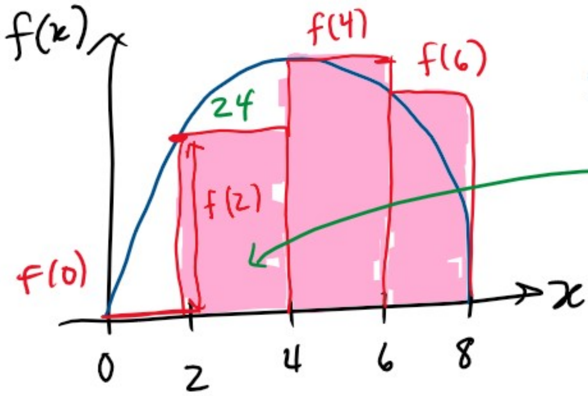
Approximate area with rectangles:

... cut into

Approximate area with rectangles

Partition: the interval $[0,\infty]$ cut into $n = 4$ equal subintervals





$f(x) = x(8-x)$

area of one rectangle

$= \text{height} \times \text{width}$

$= f(2) \times \Delta x$

$= 2(8-2) \times (2)$

$= 12 \times 2$

$= 24$

Add up all the rectangles

$$A \approx f(0)\Delta x + f(2)\Delta x + f(4)\Delta x + f(6)\Delta x$$

$$= (0)(2) + (12)(2) + (16)(2) + 12(2)$$

$$= 0 + 24 + 32 + 24 = 80 \approx A$$

This is called the left Riemann sum with $n = 4$ intervals: $\boxed{L_4 = 80}$

Observation: As n gets larger

Δx gets smaller

Δx gets smaller
and approximation of A is better

left Riemann sum with $n=8$ intervals L_8

$\Delta x = 1$

$L_8 = f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + \dots + f(7)\Delta x$

$= (0)(1) + (7)(1) + (12)(1) + (15)(1) + (16)(1) + (15)(1) + (12)(1) + (7)(1)$

$$\boxed{L_8 = 84}$$

Similarly

$L_{16} = 85$

$L_{32} = \frac{341}{4} = 85.25$

True value of A

$A = \frac{256}{3} \approx 85.3$

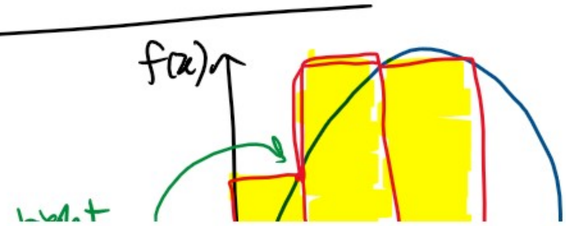
(we learn to calculate later)

GOAL: (next class)

$$\lim_{n\to\infty} L_n = A$$

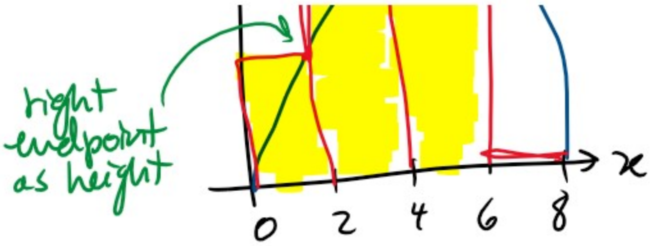
Other types of Riemann sums:

- right endpoint:

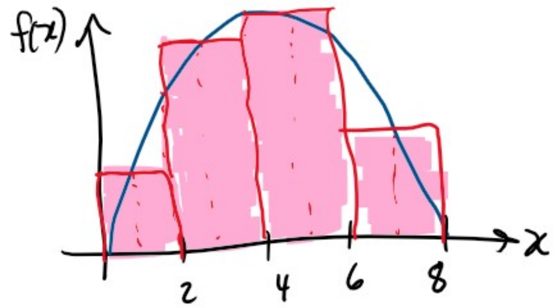


- right endpoint:

$R_4 = f(2)\Delta x + f(4)\Delta x + f(6)\Delta x + f(8)\Delta x$



$R_4 = 80$

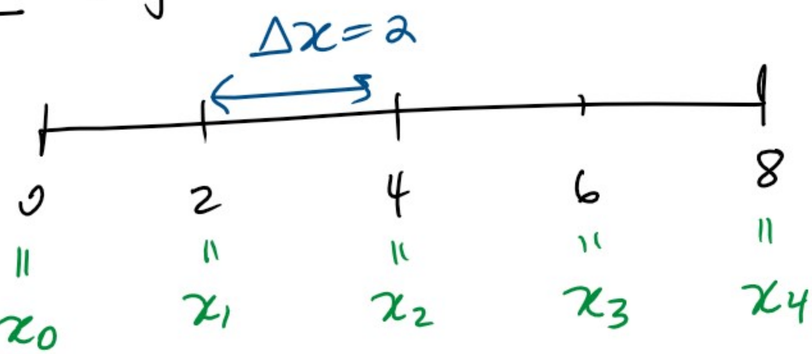


- midpoint:

$M_4 = 88$
$= f(1)\Delta x + f(3)\Delta x + f(5)\Delta x + f(7)\Delta x$

<u>Notatation:</u> sigma notation for L_n

$n=4$



$L_4 = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$

$$= \sum_{k=0}^{3} f(x_k)\Delta x$$

3 ← stop

start at $k=0$

$\sum$ says sum up all terms inside

$$= \left(f(x_0)\Delta x \right) + \left(f(x_1)\Delta x \right)$$

$$\underbrace{}_{k=0} \qquad \underbrace{}_{k=1}$$

$$+ \underbrace{\left(f(x_2)\Delta x \right)}_{k=2} + \underbrace{\left(f(x_3)\Delta x \right)}_{k=3}$$

In general for any n

$$L_n = \sum_{k=0}^{n-1} f(x_k)\Delta x$$

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x$$

← use right endpoints

$$M_n = \sum_{k=0}^{n} f\underbrace{\left(\frac{x_k + x_{k+1}}{2}\right)}_{\text{use the midpoint}} \Delta x$$

GOAL: $\displaystyle \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x = A$

Sigma Notation:

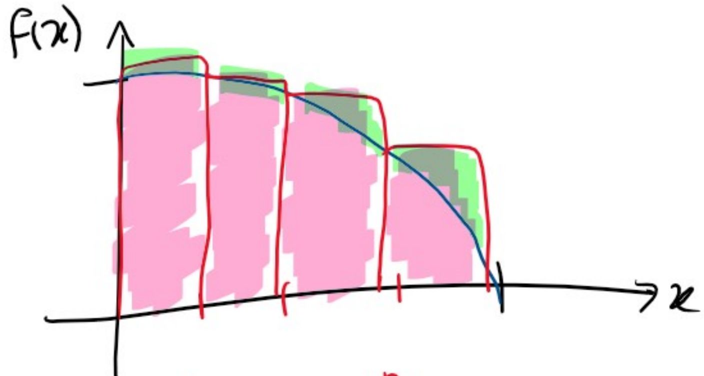
① $$\sum_{k=0}^{4} \frac{1}{k+2} = \frac{1}{0+2} + \frac{1}{1+2} + \frac{1}{2+2} + \frac{1}{3+2} + \frac{1}{4+2}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

② $$3^2 + 4^2 + 5^2 + 6^2 = \sum^{6} k^2$$

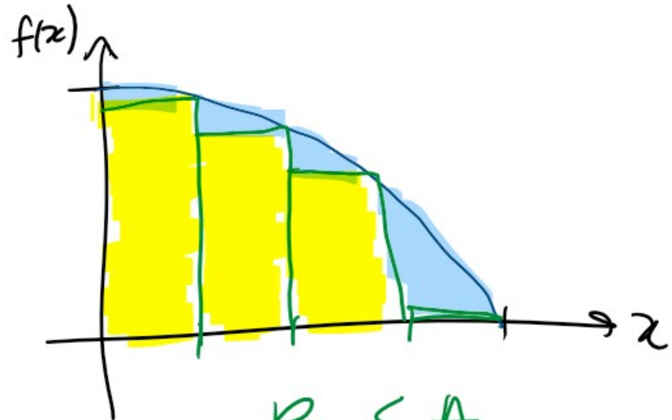
② $3^2 + 4^2 + 5^2 + 6^2 = \sum_{k=3} k$

Compare Riemann sums for $f(x)$ **decreasing**



$L_4 > A$

overestimates area



$R_4 < A$

underestimates area