Lesson 31 - LEC 100 - 11/11/2022

Definite Integrals

Warm Up → Announcements

Sum:

$$\sum_{k=0}^{5} k = 15 = \frac{5\cdot 6}{2}$$

$$\sum_{k=0}^{6} k = \sum_{k=0}^{5} + 6 = 15+6 = 21 = \frac{6\cdot 7}{2}$$

$$\sum_{k=0}^{7} k = \sum_{k=0}^{6} + 7 = 21+7 = 28 = \frac{7\cdot 8}{2}$$

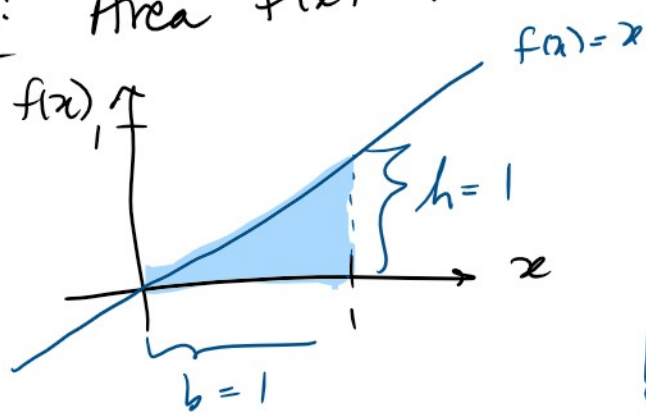
In general $\boxed{\sum_{k=0}^{n-1} k = \frac{(n-1)n}{2}}$

Last Class: Left Riemann sum L_n

Claim: $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x = A$

area under graph $f(x)$

Ex: Area $f(x) = x$ on $[0,1]$



Geometry

$A = \frac{1}{2}\cdot b \cdot h$

$= \frac{1}{2}(1)(1)$

$\boxed{A = \frac{1}{2}}$

Approximate L_4 $\qquad n = 4$

$$L_4 = \sum_{k=0}^{3} f(x_k)\Delta x$$

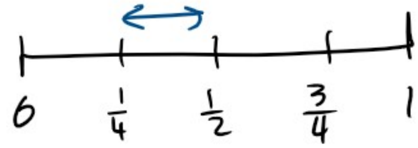
$$= \sum_{k=0}^{3} f\left(\frac{k}{4}\right)\left(\frac{1}{4}\right)$$

$$= \sum_{k=0}^{3} \left(\frac{k}{4}\right)\left(\frac{1}{4}\right)$$

$$= \frac{1}{4^2}\left(\sum_{k=0}^{3} k\right)$$

$$= \frac{6}{4^2} = \boxed{\frac{3}{8} = L_4}$$

$\Delta x = \frac{1}{4}$



$x_0 = 0$

$x_1 = \frac{1}{4} = \Delta x$

$x_2 = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = 2\Delta x$

$\vdots$

$x_k = k\Delta x = \frac{k}{4}$

$f(x) = x \qquad f\left(\frac{k}{4}\right) = \frac{k}{4}$

$$\sum_{k=0}^{n-1} k = \frac{(n-1)n}{2}$$

here $n-1 = 3$ $\Rightarrow \frac{3 \cdot 4}{2} = 6$

$n = 4$

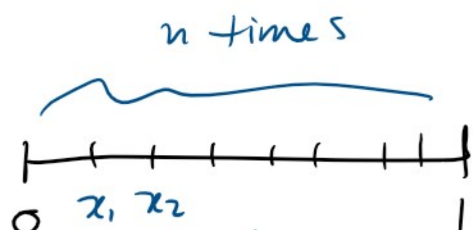
For any n

$$L_n = \sum_{k=0}^{n-1} f(x_k)\Delta x$$

$$= \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n^2}\left(\sum_{k=0}^{n-1} k\right)$$

$$= \frac{1}{n^{\cancel{2}}}\left(\frac{(n-1)\cancel{n}}{2}\right) = \frac{1}{2}\left(\frac{n-1}{n}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$



$\Delta x = \frac{1}{n}$

$x_k = k\Delta x = \frac{k}{n}$

$f(x_k) = x_k = \frac{k}{n}$

$$\sum_{k=0}^{n-1} k = \frac{(n-1)(n)}{2}$$

$$L_n = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

Claim: $\lim_{n\to\infty} L_n = A = \frac{1}{2}$ ✓

$$\lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2}$$

Left Riemann sum goes to A as $n \to \infty$

Notation:

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x = \int_a^b f(x)\,dx$$

is the <u>Definite Integral</u> of $f(x)$ over $[a,b]$

Ex: $\int_0^1 x\,dx = \lim_{n\to\infty} L_n = \frac{1}{2}$

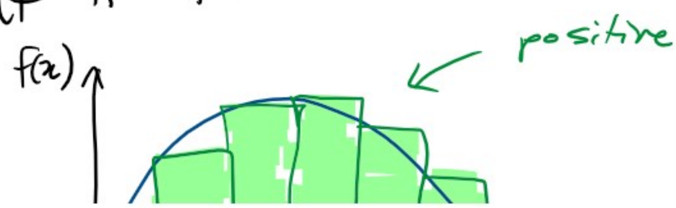
end of interval

start of interval

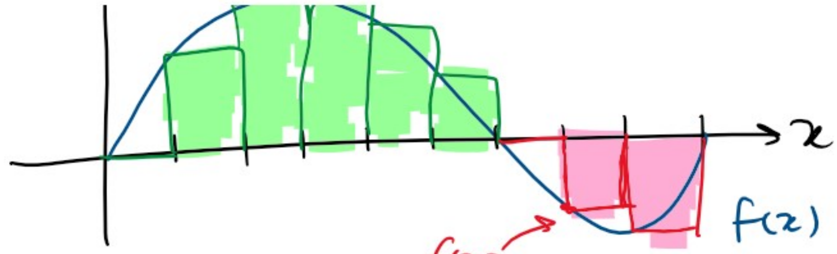
represents the area between $f(x) = x$ and the x-axis over $[0,1]$

Q: What if $f(x)$ has negative values?

Ex:



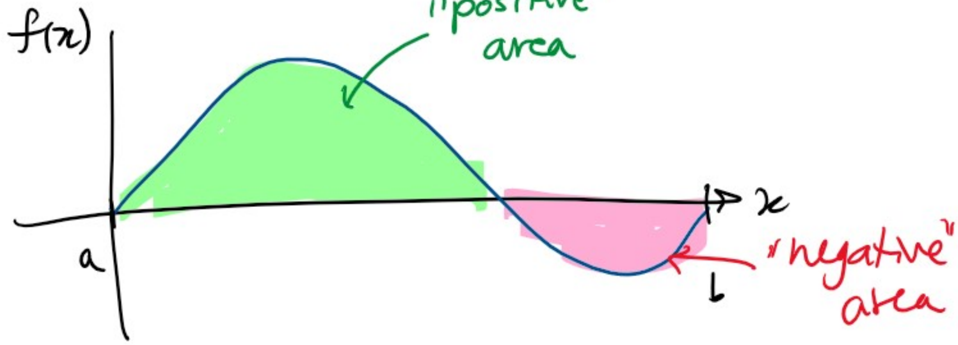
Ex:



$$L_n = \underbrace{\sum_{k=0}^{n-1} f(x_k)\Delta x}$$

when $f(x_k) < 0$
this term is negative

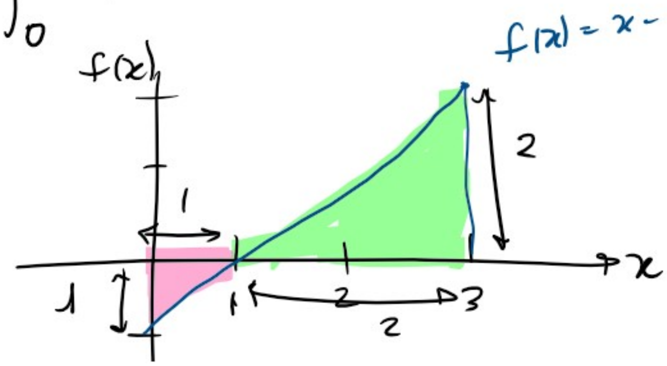
Net Area



Net Area $= \int_a^b f(x)\,dx =$ net area between the curve f(x) and the x-axis over [a, b]

Properties of the Definite Integral:

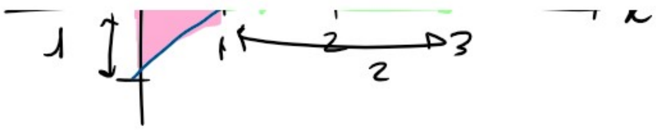
$$\int_0^3 (x-1)\,dx =$$



Geometry:

$A = -(\ \ ) + (\ \ )$

$= -\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2)$



$$= -\tfrac{1}{2}(1)(1) + \tfrac{1}{2}(2)(2)$$

$$= -\tfrac{1}{2} + 2 = \boxed{\tfrac{3}{2} = A}$$

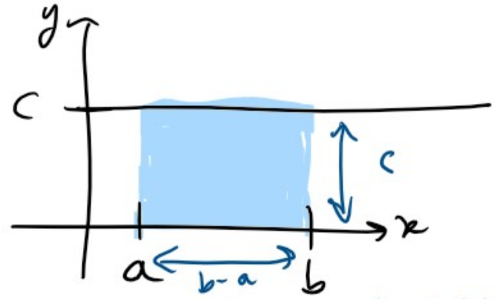
$$\int_0^3 (x-1)\,dx = \int_0^1 (x-1)\,dx + \int_1^3 (x-1)\,dx$$

In general:

① $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$a < c < b$$

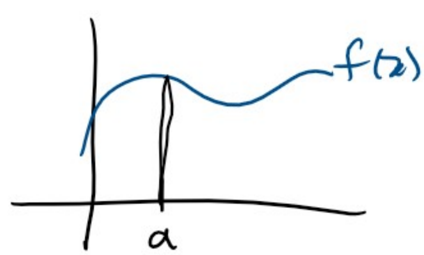
② $$\int_a^b c\,dx = c(b-a)$$



A = base × height

$= (b-a)\,c$

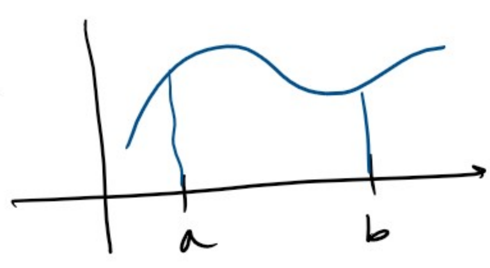
③ $$\int_a^a f(x)\,dx = 0$$



A = base × height

$= 0 \cdot f(a) = 0$

④ $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$



$$-\left(\int_a^b f(x)\,dx\right) + \int_b^a f(x)\,dx \overset{①}{=} \int_a^a f(x)\,dx \overset{③}{=} 0 - (\ )$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

⑤ $$\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

⑥ $$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

Ex: Let $I = \int_2^4 x\,dx = 6$

What is $\int_2^4 (3x-1)\,dx = ?$

$$\overset{⑤}{=} \int_2^4 3x\,dx + \int_2^4 (-1)\,dx$$

$$\overset{⑥}{=} 3\underbrace{\int_2^4 x\,dx}_{I=6} + \overset{②}{(-1)(4-2)}$$

$$= 3\cdot 6 - 2 = 18 - 2 = \boxed{16}$$