☆ Fundamental Theorem of Calculus:

Q: How is the antiderivative $F(x) = \int f(x)\,dx$ related to net area under a curve $\int_a^b f(x)\,dx$?

Intuition:

Derivative:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

- subtract off f
- divide by Δx
- $\Delta x \to 0$

Net Area:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x$$

$$= \lim_{\Delta x\to 0} \sum_{k=0}^{n-1} f(x_k)\Delta x$$

- multiply by Δx
- add up f
- $\Delta x \to 0$

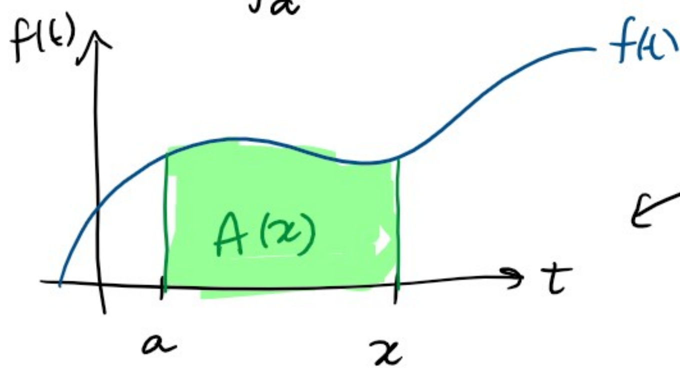
steps undo each other

More formally:

$$\text{Let } A(x) = \int_a^{x} f(t)\,dt$$

x ← variable

t is a dummy variable (keep it separate from x)

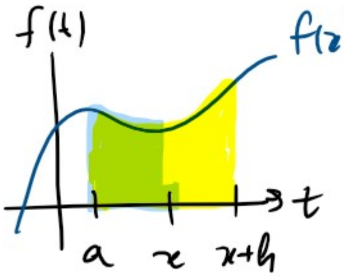


x is a variable so it can change

$A(x)$ net area under $f(t)$ over $[a, x]$
a function of x.

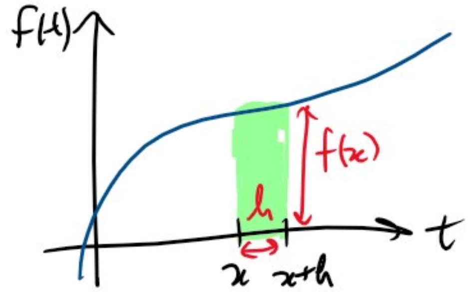
<u>Q</u>: What is $A'(x)$? The rate of change of the area over $[a, x]$

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$



$$= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$



when h is small
$h \to 0$

$A \approx f(x)h$

more accurate as $h \to 0$

$$\approx \lim_{h \to 0} \frac{1}{\cancel{h}} f(x)\cancel{h} = f(x)$$

$$A'(x) = f(x)$$

$A(x)$ is an <u>antiderivative</u> of $f(x)$

Fundamental Theorem of Calculus I

If $f(x)$ is continuous

If $f(x)$ is continuous

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

"The rate of change of the area under $f(x)$ is equal to the function $f(x)$"

<u>Ex:</u> $\frac{d}{dx}\int_x^1 (t^2-1)\,dt$ need x in the upper limit to use FTC I.

$$= \frac{d}{dx}\left(-\int_1^x (t^2-1)\,dt\right]$$

$$= \frac{d}{dx}\int_1^x (-t^2+1)\,dt \overset{FTC}{=} \boxed{-x^2+1}$$

<u>Ex:</u> Let $A(x) = \int_2^{x^2} e^t\,dt$ want only a single variable (x) to use FTC

Want $A'(x)$ use chain Rule

$A'(x) = \frac{d}{dx}A(x) = \frac{d}{dx}\left(\int_2^{x^2} e^t\,dt\right)$ Let $u = x^2$

Function of u

Chain Rule First then FTC

$$\frac{dA}{dx} = \frac{dA}{du}\cdot\frac{du}{dx}$$

$$= \left(\frac{d}{du}\int_2^u e^t\,dt\right)\cdot\frac{du}{dx}$$

$$\overset{FTC}{=} \left(e^u\right)\cdot\frac{du}{dx} = e^{x^2}(2x) = \boxed{2xe^{x^2}}$$

Q: How can we use FTC to calculate definite integrals $\int_a^b f(x)dx$?

Ex: What is $\int_0^2 (3+t)dt = ?$

WARM UP: Geometry $= 8$

Let $A(x) = \int_0^x (3+t)dt$. Want $A(2) = \int_0^2 (3+t)dt$

We know from FTC: $A'(x) = f(x) = 3+x$

$A(x)$ is an antiderivative of $3+x$

so $A(x) = F(x) + C$ ← constant of integration

What is C?

$$A(0) = \int_0^0 (3+t)dt = 0 = F(0) + C$$

$$\boxed{C = -F(0)}$$

$$A(2) = \int_0^2 (3+t)dt = F(2) + C$$

$$= F(2) - F(0)$$

Antiderivative $3+t$

$$F(t) = 3t + \frac{t^2}{2} + C$$

$$= \left(3\cdot 2 + \frac{2^2}{2}\right)^{+C} - \left(3\cdot 0 + \frac{0^2}{2}\right)^{+C}$$

$$= 6 + 2 - 0 = \boxed{8 = A(2)}$$

$$= 6 + 2 - 0 \quad = \boxed{8 = A(2)}$$

$$\boxed{\begin{array}{l} \text{Fundamental Theorem of Calculus II} \\ \displaystyle\int_a^b f(x)\,dx = F(b) - F(a) \\ \qquad \text{where } F'(x) = f(x) \\ \qquad (\text{need } f(x) \text{ is continuous on } [a,b]) \end{array}}$$

Use FTC II to calculate definite integrals.

__Ex:__ Find $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2(t)\,dt = \Big[\tan(t)\Big]_0^{\frac{\pi}{4}}$

$$= \tan\left(\frac{\pi}{4}\right) - \tan(0)$$

$$= 1 - 0 \quad = 1$$

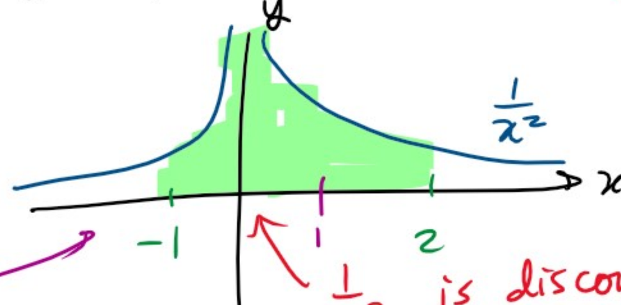
__Ex:__ $\displaystyle I = \int_{-1}^{2} \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{2} = \left[\frac{-1}{x}\right]_{-1}^{2}$

$$= \frac{-1}{2} - \left(\frac{-1}{-1}\right) = \frac{-1}{2} - 1 = -\frac{3}{2}$$

negative

$\frac{1}{x^2} > 0$

$I \geq 0$



$\frac{1}{x^2}$ is discontinuous @ $x = 0$

FTC does not apply

improper ~~singular~~ integrals

(learn in 162)

integrals
(learn in 162)

FTC does not apply

$\int_1^2 \frac{1}{x^2}\,dx$ ✓

$\int_{-2}^{-1} \frac{1}{x^2}\,dx$ ✓

Ex: $\int_0^{\frac{\sqrt{3}}{2}} \frac{-1}{\sqrt{1-x^2}}\,dx = \Big[\cos^{-1}(x)\Big]_0^{\frac{\sqrt{3}}{2}}$

$= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) - \cos^{-1}(0)$

$= \frac{\pi}{6} - \frac{\pi}{2} = \frac{\pi - 3\pi}{6} = \frac{-2\pi}{6} = \boxed{\frac{-\pi}{3}}$
