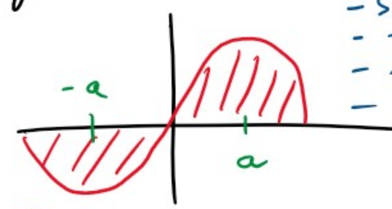
Working with Integrals & Substitution Rule

Warm Up → Announcements

Working with Integrals:

We can use symmetry to simplify definite integrals



- $\sin(x)$
- x
- x^3
- $\tan(x)$

Warm up: $\sin(-x) = -\sin(x)$

antisymmetric across $x=0$

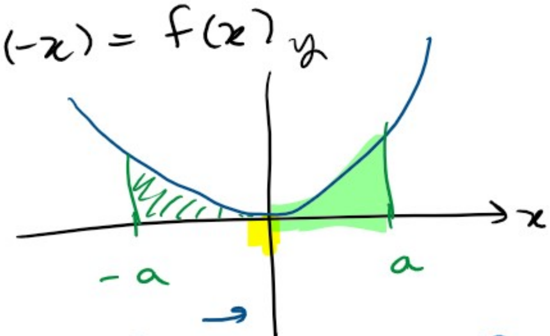
We say $f(x)$ is <u>odd</u> if
$$f(-x) = -f(x)$$

$$\boxed{\text{If } f(x) \text{ is odd, then } \int_{-a}^{a} f(x)\,dx = 0}$$

Similarly:

We say $f(x)$ is <u>even</u> if $f(-x) = f(x)$

- $\cos(x)$
- 1
- x^2
- x^4
- ⋮



mirror symmetry across $x=0$

$$\boxed{\text{If } f(x) \text{ is even, then } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

<u>Ex:</u> $\displaystyle\int_{-2}^{2} x^4\,dx = 2\int_{0}^{2} x^4\,dx$

$\overset{FTC}{=} 2\left[\dfrac{x^5}{\ }\right]^2$

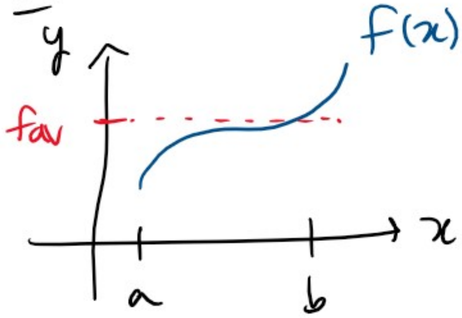
$f(x) = x^4$
$f(-x) = (-x)^4$
$= (-x)^2(-x)^2$
$= x^4$ even

FTC
$$= 2\left[\frac{x^5}{5}\right]_0^2$$
$$= 2\left[\frac{2^5}{5} - \frac{0^5}{5}\right] = \frac{2^6}{5} = \boxed{\frac{64}{5}}$$

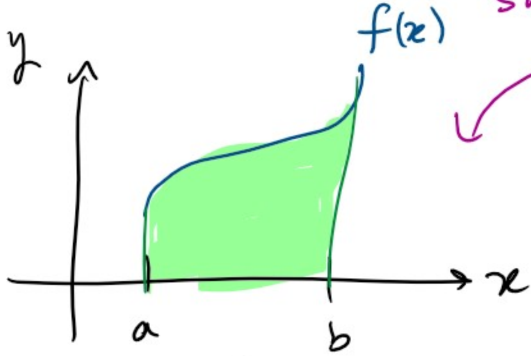
Q: What is the average value of $f(x)$ on $[a,b]$?

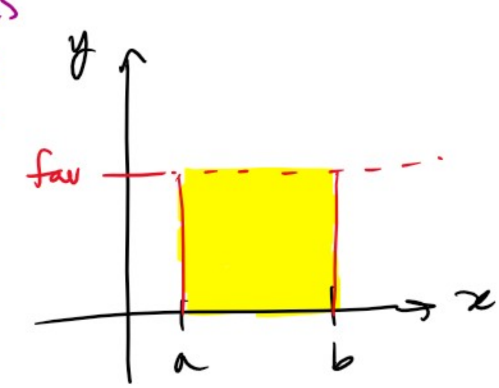
A: use integrals

Idea: use integrals + the area under curve



these areas should be equal





$$\int_a^b f(x)\,dx = (b-a)f_{av}$$

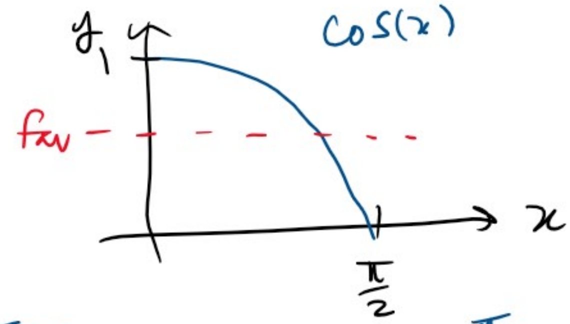
$$f_{av}(b-a) = \int_a^b f(x)\,dx$$

average value of $f(x)$ over $[a,b]$

$$\boxed{f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Ex: Let $f(x) = \cos(x)$ on $[0, \frac{\pi}{2}]$

Find f_{av}:



$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \cos(x)\,dx \overset{FTC}{=} \frac{2}{\pi}\Big[\sin(x)\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right]$$

$$= \frac{2}{\pi}\left[1 - 0\right]$$

$$\boxed{f_{av} = \frac{2}{\pi}}$$

☆ **Substitution Rule for Integrals:**

Q: What is the "Chain Rule" for antiderivatives?

A: Substitution Rule $\rightarrow$ u-substitution

Substitution Rule:

$$\int \underbrace{(x^2+3)}_{u}{}^4 \underbrace{(2x)\,dx}_{du}$$

$u = x^2 + 3$

$du = \frac{d}{dx}(x^2+3)$

$= 2x \cdot dx$

$\int u^4 \ldots = u^5 \ldots + C = \boxed{(x^2+3)^5 \ldots + C}$

$$= \int u^4 \, du = \frac{u^5}{5} + C = \boxed{\frac{(x^2+3)^5}{5} + C}$$

Q: How do you know what to make u?

Try:
- term with highest power
- most complicated term
- guess and check

Ex: $\int x^3 \cos(x^4+2)\, dx$

$u = x^4 + 2$

$du = 4x^3\, dx$

$\frac{du}{4} = x^3\, dx$

$$= \int \cos(u) \frac{du}{4}$$

$$= \frac{1}{4}\int \cos(u)\, du = \frac{1}{4}\sin(u) + C$$

$$= \boxed{\frac{1}{4}\sin(x^4+2) + C}$$

This works with definite integrals too

$$\int_0^1 (x^2+3)^4 (2x)\, dx$$

$u = x^2 + 3$

$du = 2x\, dx$

$@\, x=0 \quad u = (x^2+3)\big|_{x=0} = 3$

$@\, x=1 \quad u = (x^2+3)\big|_{x=1} = 4$

$$= \int_{x=0}^{x=1} u^4\, du$$

$\overset{FTC}{=} \left[u^5 \right]^4 \quad 4^5 \quad 3^5 \quad 781$

$x=0$

$$= \int_3^4 u^4\,du \overset{FTC}{=} \left[\frac{u^5}{5}\right]_3^4 = \frac{4^5}{5} - \frac{3^5}{5} = \frac{781}{5}$$

Ex:

$$\int_0^{\pi} \frac{\sin(\sqrt{x}+1)}{2\sqrt{x}}\,dx$$

$u = \sqrt{x} + 1$

$du = \frac{1}{2}x^{-1/2}\,dx$

$= \frac{dx}{2\sqrt{x}}$

@ $x=0$ $\quad u = \sqrt{0}+1 = 1$

@ $x=\pi$ $\quad u = \sqrt{\pi}+1$

$$= \int_1^{\sqrt{\pi}+1} \sin(u)\,du$$

$$\overset{FTC}{=} \left[-\cos(x)\right]_1^{\sqrt{\pi}+1} = \boxed{-\cos(\sqrt{\pi}+1) + \cos(1)}$$
