* typically compounded monthly

Exponential Growth & Decay

Warm Up → Announcements

Ex: Alice wants to save for a new phone worth \$1000. She puts \$50 in a savings account that has that is compounded continuously*. The interest rate is 3% per year. How long will it take for Alice to save \$1000.

Let $y(t)$ be \$ in savings account
t - time in years

@ $t=0$ $\qquad y(0) = \$50$

Want: t when $y(t) = \$1000$

interest rate proportional to the account balance

$$\frac{dy}{dt} = k\, y$$

$$\underbrace{\frac{dy}{dt}}_{\text{growth rate}} = \underbrace{(0.03)}_{\substack{3\%\text{ interest rate} \\ \left(\frac{3}{100}\right)}} \underbrace{y}_{\text{current balance}}$$

So $y(t) = A e^{0.03t}$ account has exponential growth

Q: What is A?

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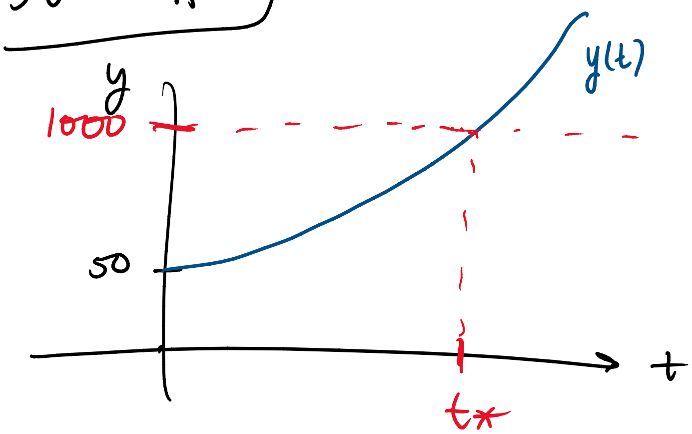
A: We know @ $t=0$ $\quad y(0) = 50$

$$y(0) = 50 = \left[A e^{0.03t} \right] \Big|_{t=0}$$

$$= A e^{0}$$

$$\boxed{50 = A}$$

$$y(t) = 50 e^{0.03t}$$



Want t_* when $y(t_*) = 1000$

$$\frac{50 e^{0.03 t_*}}{50} = \frac{1000}{50}$$

$$\ln\left(e^{0.03 t_*} = 20 \right)$$

$$0.03\, t_* = \ln(20)$$

$$t_* = \frac{\ln(20)}{0.03} \approx \boxed{99.9 \text{ years}}$$

Exponential Growth:

$$y(t) = y_0 e^{kt}$$

growth rate

$$y(t) = y_0 e^{kt}$$

growth rate
(interest rate
birth rate)

quantity that's growing
- \$
- population

initial quantity
$y(0)$

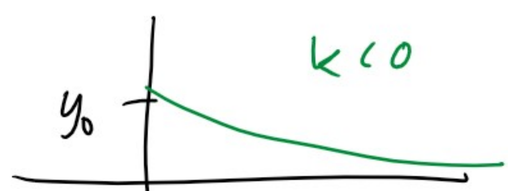
Notice that

$$\underbrace{\frac{y'(t)}{y(t)}}_{\text{relative growth rate}} = \frac{\cancel{y_0 e^{kt}} \cdot k}{\cancel{y_0 e^{kt}}} = k \quad \text{constant}$$

$$\boxed{\text{Exponential growth means } \frac{y'}{y} = k \text{ constant}}$$

Q: What if growth rate $k < 0$?

A: Exponential decay $\quad y(t) = y_0 e^{kt}$



Ex: The population of a town is decreasing at a rate of 5% per year.
... population is 10000

at a rate of 5% per year.
In 2000, the population is 10000

Then $k = -0.05$

$y_0 = 10000$

$y(t) = 10000 e^{-0.05t}$

exponential decay

<u>Def</u>: The <u>half-life</u> of an exponential decay function $y(t)$ is the time for y to decrease by $\frac{1}{2}$.

$$T_{1/2} = \frac{\ln(\frac{1}{2})}{k}$$

<u>Ex</u>: The half-life of C-14 is 5730 years used to date objects

Suppose an object has 30% of its C-14 remaining. How old is the object?

$y(t) = y_0 e^{kt}$

$T_{1/2} = 5730$

$\frac{\ln(\frac{1}{2})}{k} = 5730$

$\frac{\ln(\frac{1}{2})}{5730} = k$

$\frac{\ln(\frac{1}{2})}{5730} t$

$\frac{5730}{}$

$$\ln\left(y(t) = \cancel{y_0} e^{\frac{\ln(\frac{1}{2})}{5730} t} = 0.3 \cancel{y_0} \right)$$

$$\frac{\ln(\frac{1}{2})}{5730} t = \ln(0.3)$$

$$t = 5730 \cdot \frac{\ln(0.3)}{\ln(\frac{1}{2})} \approx \boxed{9950 \text{ years}}$$

Ex: The population of a town is 20 mil in 2000 and 22 mil in 2010

Q: what will be the population in 2025 assuming exponential growth?

$$y(t) = y_0 e^{kt}$$

let $t = 0$ 2000

$t = 10 \to 2010$

measure y in millions of ppl

$t = y - 2000$

$$y(0) = 20 = y_0 e^{k \cdot 0}$$

$$\boxed{y_0 = 20}$$

$$y(10) = 22 = 20 e^{k \cdot 10}$$

$$\ln\left(\frac{11}{10} = \frac{22}{20} = e^{10k} \right)$$

$$\ln\left(\frac{11}{10}\right) = 10k$$

$$k = \frac{\ln(\frac{11}{10})}{}$$

$$\boxed{k = \frac{\ln\left(\frac{11}{10}\right)}{10}}$$

So in 2025

$$y(25) = 20 \cdot e^{\left[\frac{\ln\left(\frac{11}{10}\right)}{10}\right]25} \approx \boxed{25.4 \text{ million}}$$