5. Compute $\int \frac{x^3}{(x-1)^2}dx$. ⟵ degree x^3 > degree $(x-1)^2$

polynomial division

A. $\frac{x^2}{2} - 2x + 3\ln|x-1| + \frac{1}{x-1} + C$

B. $\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$

C. $x + 2 + \frac{3}{x-1} + \frac{1}{(x-1)^2} + C$

D. $\frac{3(x-1)^2}{2} - \frac{1}{x-1} + C$

E. $\frac{x^4}{4} - \frac{1}{x-1} + C$

$$(x-1)^2 = x^2 - 2x + 1$$

$$\begin{array}{r} x + 2 \\ x^2 - 2x + 1 \enclose{longdiv}{x^3} \\ -(x^3 - 2x^2 + x) \\ \hline 0 + 2x^2 - x \\ -(2x^2 - 4x + 2) \\ \hline 0 + 3x - 2 \end{array}$$

so:

$$\frac{x^3}{(x-1)^2} = x + 2 + \frac{3x-2}{(x-1)^2}$$

$$\int \frac{x^3}{(x-1)^2}dx = \int x + 2 + \frac{3x-2}{(x-1)^2}\,dx$$

$$= \frac{x^2}{2} + 2x + \int \frac{3x-2}{(x-1)^2}dx$$ ⟵ Partial Fractions

$$\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$3x - 2 = A(x-1) + B$$

@ $x = 1$ $\quad 3\cdot 1 - 2 = 3\cdot 0 + B$

$$\boxed{1 = B}$$

@ $x = 0$ $\quad 3\cdot 0 - 2 = A\cdot(0-1) + 1$

$$-2 = -A + 1$$

$$A = 1 + 2 = 3 \qquad \boxed{A = 3}$$

$$= \frac{x^2}{2} + 2x + \int \frac{3}{x-1}dx + \int \frac{1}{(x-1)^2}dx$$

$u = x-1$, $du = dx$ $\qquad$ $u = x-1$, $du = dx$

$$= \frac{x^2}{2} + 2x + 3\int \frac{du}{u} + \int \frac{du}{u^2}$$

$$= \frac{x^2}{2} + 2x + 3\ln|u| + \frac{u^{-1}}{-1}$$

$$= \boxed{\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}}$$

$\boxed{B}$

3. What substitution to use to evaluate $\int \sqrt{3+3x+x^2}\,dx$?

A. $3x+\sqrt{3}=\sec t$

B. $\frac{2x}{\sqrt{3}}+\sqrt{3}=\tan t$

C. $2x+\frac{1}{\sqrt{3}}=\sin t$

D. $3x-\sqrt{3}=\tan t$

E. $2x-\sqrt{3}=\sin t$

This looks like a trig sub, but need to complete the square first

$$3+\cancel{3x}+\cancel{x^2} = \left(x+\tfrac{3}{2}\right)^2 + b$$

$$= \cancel{x^2}+\cancel{3x}+\tfrac{9}{4}+b$$

$$\tfrac{3}{4} = \tfrac{12-9}{4} = 3-\tfrac{9}{4} = b$$

$$\int \sqrt{\left(x+\tfrac{3}{2}\right)^2+\tfrac{3}{4}}\,dx$$

First do a u-sub

$u = x+\frac{3}{2}$

$du = dx$

$$= \int \sqrt{u^2+\tfrac{3}{4}}\,du$$

Then do a trig sub

$u = \frac{\sqrt{3}}{2}\tan\theta$

combine the 2 substitutions together

$$\tfrac{\sqrt{3}}{2}\tan\theta = u = x+\tfrac{3}{2}$$

solve for $\tan\theta$

$$\tan\theta = \tfrac{2}{\sqrt{3}}\left(x+\tfrac{3}{2}\right) = \tfrac{2x}{\sqrt{3}}+\sqrt{3}$$

$\boxed{B}$

5. $\displaystyle\int_{\frac{1}{12}}^{\frac{1}{4}} \frac{12\,dt}{\sqrt{t}+4t\sqrt{t}} =$

A. 6π

B. 4π

C. π

D. 3π

E. 2π

Let's do a u-sub to get rid of the $\sqrt{t}$

$u = \sqrt{t}$

$du = \frac{1}{2}t^{-1/2}\,dt \qquad t^{-1/2}dt = 2du$

$$\int_{\frac{1}{2\sqrt{3}}}^{\frac{1}{2}} \frac{12}{1+4u^2}\,(2du)$$

$$= 24\int_{\frac{1}{2\sqrt{3}}}^{\frac{1}{2}} \frac{(du)\ \frac{1}{4}}{(1+4u^2)\ \frac{1}{4}}$$

$$= 6\int_{\frac{1}{2\sqrt{3}}}^{\frac{1}{2}} \frac{du}{\frac{1}{4}+u^2}$$

Trig sub

$u = \frac{1}{2}\tan\theta$

$du = \frac{1}{2}\sec^2\theta\,d\theta$

$\frac{1}{2}\tan\theta = \frac{1}{2} \to \theta = \frac{\pi}{4}$

$\frac{1}{2}\tan\theta = \frac{1}{2\sqrt{3}} \quad \theta = \frac{\pi}{6}$

$$= 6\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{4}+\left(\frac{1}{2}\tan\theta\right)^2} = 6\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cancel{\frac{1}{2}}\,\cancel{\sec^2\theta}\,d\theta}{2\,\cancel{\frac{1}{4}}\,\cancel{\sec^2\theta}}$$

$$= 12\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta = \Big[12\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 12\cdot\tfrac{\pi}{4} - 12\tfrac{\pi}{6}$$

$$= 3\pi - 2\pi = \boxed{\pi}$$

$\boxed{C}$

7. Evaluate the integral

$$\int_{\frac{1}{3}}^{\frac{\sqrt{3}}{3}} \frac{dx}{x^2\sqrt{9x^2+1}}.$$

← Looks like trig sub

A. $3\sqrt{2}-2\sqrt{3}$
B. $\sqrt{2}-\frac{2\sqrt{3}}{3}$
C. $\frac{\sqrt{3}}{2}-\frac{1}{2}$
D. $\sqrt{3}-1$
E. $\sqrt{2}-1$

$$\int_{\frac{1}{3}}^{\frac{\sqrt{3}}{3}} \frac{dx}{3x^2\sqrt{\frac{9x^2+1}{9}}} = \frac{1}{3}\int_{\frac{1}{3}}^{\frac{\sqrt{3}}{3}} \frac{dx}{x^2\sqrt{x^2+\frac{1}{9}}}$$

trig sub: $x = \frac{1}{3}\tan\theta$ $\quad dx = \frac{1}{3}\sec^2\theta\, d\theta$

$\frac{1}{3}\tan\theta = \frac{1}{3}$ $\quad \theta = \frac{\pi}{4}$

$\frac{1}{3}\tan\theta = \frac{\sqrt{3}}{3}$ $\quad \theta = \frac{\pi}{3}$

$$= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{1}{3}\sec^2\theta\, d\theta}{\left(\frac{1}{3}\tan\theta\right)^2\sqrt{\left(\frac{1}{3}\right)^2(\tan^2\theta+1)}}$$

$$= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cancel{\frac{1}{3}}\sec^{\cancel{2}}\theta\, d\theta}{\frac{1}{9}\tan^2\theta \cdot \cancel{\frac{1}{3}}\cancel{\sec\theta}}$$

$$= \frac{9}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\, d\theta}{\tan^2\theta} = 3\left[-\csc(x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -3\left[\csc\left(\frac{\pi}{3}\right)-\csc\left(\frac{\pi}{4}\right)\right]$$

$$= -3\left[\frac{2}{\sqrt{3}}-\sqrt{2}\right]$$

$$= \frac{-6}{\sqrt{3}}+3\sqrt{2} = \boxed{3\sqrt{2}-2\sqrt{3}} \quad \boxed{A}$$

3. Compute $\int \sin^5(x)\cos^2(x)dx$.

← Trig Integral → split off a factor of $\sin(x)$

A. $-\frac{1}{3}\cos^2(x)+\frac{2}{5}\cos^4(x)-\frac{1}{7}\cos^6(x)+C$
B. $-\frac{1}{3}\cos^3(x)+\frac{2}{5}\cos^4(x)-\frac{1}{7}\cos^6(x)+C$
C. $-\frac{1}{3}\cos^3(x)+\frac{2}{5}\cos^5(x)-\frac{1}{7}\cos^6(x)+C$
D. $-\frac{1}{3}\cos^3(x)+\frac{2}{5}\cos^5(x)-\frac{1}{7}\cos^7(x)+C$
E. $\frac{1}{3}\cos^2(x)-\frac{2}{5}\cos^5(x)+\frac{1}{7}\cos^7(x)+C$

$$\int \sin^4(x)\cos^2(x)\sin(x)\,dx$$

$u = \cos(x)$
$du = -\sin(x)\,dx$

$$\int \left[1-\cos^2(x)\right]^2\cos^2(x)\sin(x)\,dx$$

$$= \int \left[1-u^2\right]^2 u^2\,(-du)$$

$$= \int -(1-2u^2+u^4)u^2\,du = \int -u^2+2u^4-u^6\,du$$

$$= -\frac{u^3}{3}+\frac{2u^5}{5}-\frac{u^7}{7}+C$$

$$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C$$

$$= -\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C$$

$\boxed{D}$

9. Compute the integral
$$\int \frac{x^2+x+2}{x^2+2x+2}\,dx.$$

A. $x - \ln(x^2+2x+2) + C$

B. $x - \ln(x^2+2x+2) + \tan^{-1}(x+1) + C$

C. $x - \frac{1}{2}\ln(x^2+2x+2) + \tan^{-1}(x+1) + C$

D. $x + \ln(x^2+2x+2) - \frac{1}{2}\tan^{-1}(x+1) + C$

E. $\frac{1}{2}x\ln(x^2+2x+2) + 2\tan^{-1}(x+1) + C$

Want to use partial fractions for $\int \frac{P(x)}{Q(x)}\,dx$

but degree $P(x)$ = degree $Q(x) = 2$

so need to do polynomial division first

$$\begin{array}{r|l} & 1 \\ x^2+2x+2 & x^2+x+2 \\ & -(x^2+2x+2) \\ \hline & 0 \quad -x + 0 \end{array}$$

$$\int \frac{x^2+x+2}{x^2+2x+2}\,dx = \int 1 + \frac{-x}{x^2+2x+2}$$

↑ complete the square

$\cancel{x^2} + \cancel{2x} + 2 = (x+1)^2 + b$

$= \cancel{x^2} + \cancel{2x} + 1 + b$

$1 = b$

$$= x + \int \frac{-x\,dx}{(x+1)^2+1}$$

u-sub

$u = x+1$ $\quad x = u-1$

$du = dx$

$$= x + \int -\frac{(u-1)}{u^2+1}\,du$$

$$= x + -\int \frac{u\,du}{u^2+1} + \int \frac{du}{u^2+1}$$

another subst.

$v = u^2+1$

$dv = 2u\,du$

Trig subst.

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$$= x - \frac{1}{2}\int \frac{dv}{v} + \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta+1}$$

$= \sec^2\theta$

$$= x - \frac{1}{2}\int \frac{dv}{v} + \int d\theta$$

$$= x - \frac{1}{2}\ln|v| + \theta + C$$

Want answer in original variable x

$v = u^2+1, \quad u = \tan\theta, \quad u = x+1$

$$= x - \frac{1}{2}\ln|u^2+1| + \tan^{-1}(u) + C$$

$$= x - \frac{1}{2}\ln|(x+1)^2+1| + \tan^{-1}(x+1) + C$$

$$= x - \frac{1}{2}\ln|x^2+2x+2| + \tan^{-1}(x+1) + C \qquad \boxed{C}$$

4. After an appropriate trigonometric substitution, the integral

$$\int \frac{\sqrt{7x^2-1}}{x^2}\,dx$$

becomes

A. $\frac{1}{\sqrt{7}}\int \tan(\theta)d\theta$

B. $\frac{1}{\sqrt{7}}\int \frac{\tan^2(\theta)}{\sec(\theta)}d\theta$

C. $\sqrt{7}\int \frac{\tan^2(\theta)}{\sec(\theta)}d\theta$

D. $\frac{1}{7}\int \tan(\theta)d\theta$

E. $\int \frac{\sec^3(\theta)}{\tan^2(\theta)}\,d\theta$

$$\int \frac{\sqrt{7x^2-1}}{x^2}$$

$$= \int \frac{\sqrt{7}\sqrt{\frac{7x^2-1}{7}}}{x^2}\,dx$$

$$= \sqrt{7}\int \frac{\sqrt{x^2-\frac{1}{7}}}{x^2}\,dx$$

Trig substitution $\quad x = \frac{1}{\sqrt{7}}\sec\theta$

$dx = \frac{1}{\sqrt{7}}\tan\theta\sec\theta\,d\theta$

$$= \sqrt{7}\int \frac{\sqrt{(\frac{1}{7})(\sec^2\theta-1)}}{\frac{1}{7}\sec^{\cancel{2}}\theta}\cdot\frac{1}{\sqrt{7}}\tan\theta\,\cancel{\sec\theta}\,d\theta$$

$$= \sqrt{7}\int \frac{\frac{1}{\cancel{\sqrt{7}}}\tan\theta}{\frac{1}{\cancel{7}}\sec\theta}\cdot\frac{1}{\cancel{\sqrt{7}}}\tan\theta\,d\theta$$

$$= \sqrt{7}\int \frac{\tan^2\theta\,d\theta}{\sec\theta} \qquad \boxed{C}$$

10. Evaluate the improper integral

$$\int_0^\infty \frac{e^{3x}}{e^{6x}+1}dx.$$

$$= \lim_{b\to\infty}\int_0^b \frac{e^{3x}}{e^{6x}+1}\,dx$$

. π 2

10. Evaluate the improper integral
$$\int_0^\infty \frac{e^{3x}}{e^{6x}+1}dx.$$

A. $\frac{\pi}{2} - \frac{2}{\sqrt{3}}$
B. $\ln(1+\sqrt{3}) - 1$
C. $\frac{\pi}{4}$
D. $\frac{\pi}{12}$
E. The improper integral is divergent.

$$= \lim_{b\to\infty} \int_0^b \frac{e^{3x}}{e^{6x}+1}\,dx$$

u substitution:

$u = e^{3x}$

$du = 3e^{3x}\,dx$

$$= \lim_{b\to\infty} \frac{1}{3}\int_1^{e^{3b}} \frac{du}{u^2+1}$$

Now trig subst.

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$$= \frac{1}{3}\lim_{b\to\infty} \int_{\tan^{-1}(1)}^{\tan^{-1}(e^{3b})} \frac{\sec^2\theta\, d\theta}{\tan^2\theta + 1}$$

$$= \frac{1}{3}\lim_{b\to\infty} \int_{\tan^{-1}(1)}^{\tan^{-1}(e^{3b})} \frac{\cancel{\sec^2\theta}\, d\theta}{\cancel{\sec^2\theta}} = \frac{1}{3}\lim_{b\to\infty} \Big[\theta\Big]_{\tan^{-1}(1)}^{\tan^{-1}(e^{3b})}$$

$$= \frac{1}{3}\lim_{b\to\infty}\left[\tan^{-1}(e^{3b}) - \tan^{-1}(1)\right]$$

$$= \frac{1}{3}\lim_{b\to\infty}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{1}{3}\left[\frac{\pi}{4}\right] = \frac{\pi}{12}$$

D

5. Evaluate the following integral
$$\int_4^{11/2} \frac{1}{\sqrt{-x^2+8x-7}}dx.$$

A. $\frac{\pi}{12}$
B. $\frac{2\pi}{3}$
C. $\frac{\pi}{3}$
D. $\frac{\pi}{6}$
E. $\frac{\pi}{24}$

This looks like a trig sub

need to complete the square first

$-\cancel{x^2} + \cancel{8x} - 7 = -(x-4)^2 + b$

$= -(x^2 - 8x + 16) + b$

$= -\cancel{x^2} + \cancel{8x} - 16 + b$

$16 - 7 = b$ $\quad \boxed{b = 9}$

$$\int_4^{11/2} \frac{dx}{\sqrt{9-(x-4)^2}}$$

u-sub.

$u = x - 4$

$du = dx$

$$= \int_0^{3/2} \frac{du}{\sqrt{9-u^2}}$$

Trig subst. $u = 3\sin\theta$

$du = 3\cos\theta\, d\theta$

$0 = 3\sin\theta \to \theta = 0$

$\frac{3}{2} = 3\sin\theta \to \theta = \frac{\pi}{6}$

$$\int_0 \quad \sqrt{9-u^2}$$

$$= \int_0^{\frac{\pi}{6}} \frac{3\cos\theta \, d\theta}{\sqrt{9(1-\sin^2(\theta))}} = \int_0^{\frac{\pi}{6}} \frac{\cancel{3\cos\theta}}{\cancel{3\cos\theta}} \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} d\theta = \Big(\theta\Big]_0^{\pi/6} = \frac{\pi}{6} - 0 = \frac{\pi}{6} \qquad \boxed{D}$$

6. For what values of p is the integral $\int_0^1 \frac{1}{x^{3p+7}} \, dx$ convergent?

↑ this function is undefined at $x = 0$

A. $p \leq -1$

B. $p > -10$

C. $p \geq -2$

D. $\boxed{p < -2}$

E. $p < -3$

$$= \lim_{b \to 0^+} \int_b^1 \frac{1}{x^{3p+7}} \, dx$$

$$= \lim_{b \to 0^+} \int_b^1 x^{-3p-7} \, dx$$

$$= \lim_{b \to 0^+} \left[\frac{x^{-3p-7+1}}{-3p-7+1} \right]_b^1$$

$$= \lim_{b \to 0^+} \left[\frac{1^{-3p-6}}{-3p-6} - \frac{b^{-3p-6}}{-3p-6} \right]$$

→ this limit goes to zero as long as the exponent is greater than 0

Want $-3p-6 > 0$

$$-3p > -6$$

$$3p < 6$$

$$\boxed{p < 2} \qquad \boxed{D}$$

8. Find all values of p for which the integral $\int_1^\infty \frac{1}{x^{1-p}} dx$ converges.

I believe this problem has a typo. The correct solution is not an option

A. $p < 2$

B. $p < 1$

C. $p > 2$

D. $p > 1$

E. Diverges for all p.

$$= \lim \int^b \frac{1}{\quad} \, dx = \lim \int^b x^{p-1} \, dx$$

C. $p > 2$

D. $p > 1$

E. Diverges for all p.

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^{1-p}}\,dx = \lim_{b\to\infty} \int_1^b x^{p-1}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^p}{p}\right]_1^b = \lim_{b\to\infty} \left[\frac{b^p}{p} - \frac{1^p}{p}\right]$$

this term converges to zero only if $p<0$

Solution should be $\boxed{p<0}$

10. $\displaystyle\int_0^{\pi^2} \sin\frac{\sqrt{y}}{2}\,dy =$

A. 1

B. 2

C. 3

D. 5

E. 8

Let's do a u-sub to get rid of the $\sqrt{y}$

Let $x = \frac{\sqrt{y}}{2}$ $\quad dx = \frac{1}{2} y^{-1/2} \cdot \frac{dy}{2} = \frac{1}{4\sqrt{y}}\,dy$

$dy = 4\sqrt{y}\,dx = 4(2x)\,dx$

$$= \int_0^{\frac{\pi}{2}} \sin(x)\cdot 8x\,dx$$

$$= 8\int_0^{\frac{\pi}{2}} x\sin(x)\,dx$$

Integration By Parts

$u = x$ $\quad dv = \sin(x)\,dx$

$du = dx$ $\quad v = -\cos(x)$

$$= 8\left[u\cdot v - \int v\,du\right] = 8\left[x(-\cos(x)) - \int -\cos(x)\,dx\right]$$

$$= 8\left[-x\cos(x) + \sin(x)\right]_0^{\frac{\pi}{2}}$$

$$= 8\left[-\frac{\pi}{2}\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} + 0\cdot\underbrace{\cos(0)}_{0} - \underbrace{\sin(0)}_{0}\right]$$

$= 8$ $\quad \boxed{E}$