Lesson 10 - MA 166 - Spring 2022

8.2 Integration by Parts

Announcements:

- Class Website:
  https://www.math.purdue.edu/~kthood/MA166_Spring2022.html
  - seating chart for exam 1
  - sample cover sheet for exam 1

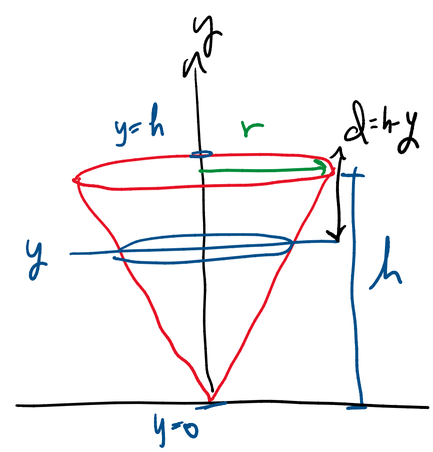
GOALS:

- evaluate integrals using Integration by Parts (IBP)
- multiple iterations of IBP

WARM UP: A water tank is shaped like an inverted cone with:

height $h = 6m$

base radius $r = 1.5m$



What is the cross sectional area $A(y)$ of the tank?

cross section

$A(y) = \pi r^2$

$@\ y=0 \quad r=0$

$@\ y=6 \quad r=1.5m$

$r = 1.5\frac{y}{6} = \frac{y}{4}$

$$\boxed{A(y) = \pi\left(\frac{y}{4}\right)^2 = \frac{\pi y^2}{16}}$$

Want: Find work to pump water out of the top of the tank

density $\rho = 1000\ kg/m^3$

gravity $g = 9.8\ m/s^2$

$$W_k = F_k \cdot d_k = mg \cdot d$$

$$= (\rho V)\cdot g \cdot d = \rho \cdot A(y)\Delta y \cdot g \cdot d$$

$$= \rho\left(\frac{\pi y^2}{4}\right)\Delta y\ g\ (h-y)$$

$$\sum W_k \longrightarrow \quad n \to \infty$$

$\left(\; 4 \; \right)$

$\sum W_k \longrightarrow \quad n \to \infty$

$$W = \int_0^6 \rho \frac{\pi y^2}{4} g (h-y)\, dy$$

$$\boxed{W = \frac{\pi}{4} \rho g \int_0^6 y^2 (6-y)\, dy}$$

II. Integration by Parts:

Product rule for derivatives : $u(x), v(x)$

$$(uv)' = u'v + uv'$$

integrate

$$\int (uv)'\, dx = \int (u'v + uv')\, dx$$

$$uv = \int u'v\, dx + \underbrace{\int uv'\, dx}$$

Rearrange

$$\int u \underbrace{v'dx}_{dv} = uv - \int u'v\, dx$$

$$\int u\, dv = uv - \int v \underbrace{u'dx}_{du}$$

$$\boxed{\begin{array}{c}\int u\,dv = uv - \int v\,du \\ \text{Integration by Parts:} \\ \text{need } u(x) \text{ and } v(x) \text{ are differentiable}\end{array}}$$

Ex: Evaluate $\int x e^x\,dx$

$$\int \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv} = \int u\,dv$$

$u = x$

$du = 1 \cdot dx$

$dv = e^x\,dx$

$v = \int dv = \int e^x\,dx = e^x$

IBP

$$= u \cdot v - \int v\,du$$

$$= x \cdot e^x - \int e^x\,dx$$

we can integrate this!

$$= xe^x - e^x + C$$

$$\boxed{\int x e^x\,dx = (x-1)e^x + C}$$

Ex: Evaluate $\int x \sin(x)\,dx$

$$\int x \sin(x)\,dx = \int u\,dv = u \cdot v - \int v\,du$$

Q: What should we take for u?

$u = x$ or $u = \sin(x)$

Try: $u = \sin(x)$

$dv = x\,dx$

$v = \int dv = \int x\,dx = \frac{x^2}{2}$

Try: $u = \sin(x)$ $\quad dv = x\,dx$

$du = \cos(x)\,dx \quad v = \int dv = \int x\,dx = \frac{x^2}{2}$

$= u \cdot v - \int v\,du$

$= \sin(x) \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cos(x)\,dx$

Try other option for u ← this integral is worse than the original

Let $u = x \quad dv = \sin(x)\,dx$

$du = dx \quad v = \int dv = \int \sin(x)\,dx = -\cos(x)$

$\int x \sin(x)\,dx = u \cdot v - \int v\,du$

$= x(-\cos(x)) - \int -\cos(x)\,dx$ this is a simpler integral

$$\boxed{\int x \sin(x)\,dx = -x\cos(x) + \sin(x) + C}$$

NOTE: In this case, it was better to choose $u = x$

In general, it is good to choose $u = x^n$ where $n \geq 1$ (if available)

Ex: Evaluate $\int \ln(x)\,dx$

IBP:

$\int \ln(x)\,dx = \int u\,dv = u \cdot v - \int v\,du$

$u = \ln(x) \quad dv = dx$

$du = \frac{1}{x} \cdot dx \quad v = \int dv = \int dx = x$

$u = \ln(x)$ $\quad\quad\quad\quad$ $v = \int dv = \int dx = x$

$du = \frac{1}{x} \cdot dx$

$= u \cdot v - \int v\, du$

$= \ln(x) \cdot x - \int \cancel{x}\, \frac{1}{\cancel{x}} \cdot dx$

$$\boxed{\int \ln(x)\,dx = x \ln x - x + C}$$

II. Multiple Iterations of IBP:

Ex: Evaluate $\int x^2 e^x\, dx$

$\int x^2 e^x\, dx = \int u\, dv = u \cdot v - \int v\, du$

$u = x^2$ $\quad\quad\quad\quad$ $dv = e^x\, dx$

$du = 2x\, dx$ $\quad\quad\quad\quad$ $v = \int dv = \int e^x\, dx = e^x$

$= u \cdot v - \int v\, du$

$= x^2 \cdot e^x - \int e^x \cdot 2x\, dx$ $\quad$ ← product of fcns (easier than original integral) ← IBP again

$= x^2 e^x - 2 \int x e^x\, dx$

$u = x$ $\quad\quad\quad\quad$ $dv = e^x\, dx$

$du = dx$ $\quad\quad\quad\quad$ $v = \int dv = \int e^x\, dx = e^x$

$= x^2 e^x - 2\left[u \cdot v - \int v\, du\right]$

$= x^2 e^x - 2\left[x e^x - \int e^x\, dx\right]$

$$= x^2 e^x - 2\left[x e^x - \int e^x \, dx \right]$$

$$= x^2 e^x - 2\left[x e^x - e^x + c_1 \right]$$

$$= x^2 e^x - 2xe^x + 2e^x + c_2$$

constant of integration unknown (different from each other $c_2 = -2c_1$)

$$\boxed{\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x + C}$$

Observations: iterated IBP $\times 2$

1st: $u = x^2$ $du = 2x\,dx$

2nd: $u = x$ $du = dx$

If possible, choose u so that eventually, $du = k\,dx$ ← constant multiple

Want choose $u = x^n$

1st $u = x^n$

2nd $u = nx^{n-1}$

$\vdots$

n^{th} $u = x$ $du = dx$

Ex: $\int x^3 \sin(x)\,dx$ ← apply IBP 3 times

1st : choose $u = x^3$ $dv = \sin(x)dx$

$du = 3x^2$ $v = \int \sin(x)dx = -\cos(x)$

$\hookrightarrow u \cdot v - \int v\,du \longrightarrow \int -\cos(x)\, 3x^2 dx$

2nd IBP: choose $u = 3x^2$, $du = 6x\,dx$ $\quad$ $dv = -\cos(x)\,dx$, $v = -\sin(x)$

$\hookrightarrow \quad u \cdot v - \int v\,du \Rightarrow \int -\sin(x)\,6x\,dx$

3rd IBP: choose $u = 6x$, $du = 6\,dx$ $\quad$ $dv = -\sin(x)\,dx$, $v = \cos(x)$

$\hookrightarrow \quad u \cdot v - \int v\,du \Rightarrow \int \cos(x)\,6\,dx$

stop IBP here

Ex: Multiple Iterations of IBP

Evaluate $\int e^{2x} \sin(x)\,dx$

$$\int e^{2x} \sin(x)\,dx = \int u\,dv = u \cdot v - \int v\,du$$

No obvious choice of u

Try: $u = e^{2x}$, $du = 2e^{2x}\,dx$ $\quad$ $dv = \sin(x)\,dx$, $v = \int dv = \int \sin(x)\,dx = -\cos(x)$

$$= u \cdot v - \int v\,du$$

$$= e^{2x} \cdot (-\cos(x)) - \int -\cos(x)\,2e^{2x}\,dx$$

$$= -e^{2x} \cos(x) + 2 \underbrace{\int e^{2x} \cos(x)\,dx}_{\text{Not easier than original integral}}$$

BUT ... keep trying

Apply IBP one more time

e^{2x} $\quad$ $dv = \cos(x)\,dx$

Apply IBP
one more time

$$u = e^{2x} \qquad dv = \cos(x)\,dx$$

$$du = 2e^{2x}\,dx \qquad v = \int dv = \sin(x)$$

$$= -e^{2x}\cos(x) + 2\left\{ u \cdot v - \int v\,du \right\}$$

$$= -e^{2x}\cos(x) + 2\left\{ e^{2x}\sin(x) - \int \sin(x)\, 2e^{2x}\,dx \right\}$$

$$\underbrace{\int e^{2x}\sin(x)\,dx} = -e^{2x}\cos(x) + 2e^{2x}\sin(x) - 4\underbrace{\int e^{2x}\sin(x)\,dx}$$

$$\frac{5\int e^{2x}\sin(x)\,dx}{5} = \frac{-e^{2x}\cos(x) + 2e^{2x}\sin(x)}{5}$$

$$\boxed{\int e^{2x}\sin(x)\,dx = \frac{1}{5}e^{2x}\left[2\sin(x) - \cos(x)\right] + C}$$

NOTE: This can happen when $v = \sin(x)$ or $\cos(x)$ because v and dv alternate between sine + cosine

So you can end up with the integral you started from.

Ex: $\int \sin^{-1}(x)\,dx$ Apply IBP

$$= \int u\,dv = u \cdot v - \int v\,du$$

$$u = \sin^{-1}(x) \qquad dv = dx$$

$u = \sin^{-1}(x)$ $\qquad dv = dx$

$du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ $\qquad v = \int dv = \int dx = x$

$$= u \cdot v - \int v\,du = \sin^{-1}(x) \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$$

u-substitution

let $u = 1-x^2$ $\quad du = -2x\,dx$

$x\,dx = \frac{-1}{2}du$

$$= x\sin^{-1}(x) - \int \left(-\tfrac{1}{2}\right) u^{-1/2}\,du$$

$$= x\sin^{-1}(x) + \cancel{\tfrac{1}{2}}\left[\frac{u^{1/2}}{\cancel{1/2}} + C\right]$$

$$\boxed{\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C}$$