8.3 - Trigonometric Integrals - Part 1

Announcements:

Extensions - Quiz 3 due Fri 2/4
- HW lesson 10 due Sat 2/5

Exam 1 is Wed Feb 9 @ 6:30pm

GOALS:

- evaluate integrals with powers of sine and cosine

WARM UP: Apply u-substitution to convert the integral to something that is suitable for IBP:

$$\int_0^{\pi^2/4} \sin(\sqrt{x})\,dx$$

$u = \sqrt{x} = x^{1/2}$ $\qquad x^{-1/2} = \frac{1}{u}$

$du = \frac{1}{2}x^{-1/2}dx$

$= \frac{1}{2u}dx$

$dx = 2u\,du$

$\boxed{\int_0^{\pi/2} \sin(u)(2u\,du)}$

(a) $\int_0^{\frac{\pi}{2}} \sin(u)\, x^{1/2}\, du$

(b) $\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(u)\frac{du}{u}$

$\boxed{\text{(c) } 2\int_0^{\pi/2} \sin(u)\, u\, du}$

I. Powers of sin(x) and cos(x):

Want to evaluate integrals:

$$\int \sin^m(x)\cos^n(x)\,dx$$

where m and n are real numbers

Several ways to do this - depends on m and n

m	n	method to solve
odd	real	?

odd ≥ 0	real number	?

Ex: $\int \sin^3(x)\cos^2(x)\,dx$

Method - split off one factor of $\sin(x)$

$$= \int \underbrace{\sin^2(x)} \cos^2(x) \underbrace{\sin(x)\,dx}_{-du}$$

rewrite in terms of $\cos(x)$

u-substitution

$u = \cos(x) \qquad du = -\sin(x)\,dx$

Trig identity: $\sin^2(x) + \cos^2(x) = 1$

$\sin^2(x) = 1 - \cos^2(x)$

$$= \int (1-\cos^2(x))\cos^2(x)\,\sin(x)\,dx$$

$u = \cos(x) \qquad du = -\sin(x)\,dx$

$$= \int (1-u^2)\,u^2\,(-du)$$

$$= \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \boxed{\frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + C}$$

m	n	method
odd >0	real #	split off a factor of $\sin(x)$ ↳ apply u-substitution

real #	odd >0	?

Ex: $\int \cos^3(x)\,dx = \int \sin^0(x)\cos^3(x)\,dx$ $m=0$

split off one factor of $\cos(x)$

$$= \int \underbrace{\cos^2(x)}_{\text{trig formula}} \underbrace{\cos(x)dx}_{du}$$

u-substitution
$u = \sin(x)$
$du = \cos(x)dx$

trig formula
$\cos^2(x) = 1 - \sin^2(x)$

$$= \int [1-\sin^2(x)] \cos(x)dx$$

$$= \int [1-u^2]\,du = u - \frac{u^3}{3} + C$$

$$= \boxed{\sin(x) - \frac{\sin^3(x)}{3} + C}$$

m	n	method
odd >0	real #	split off a factor of sine ↳ u-substitution
real #	odd >0	split of a factor of $\cos(x)$ ↳ u-substitution
even ≥0	even ≥0	?

Ex: $\int \sin^2(x)\cos^2(x)\,dx$

use half angle formulas:

$$\text{Ⓐ} \quad 1 = \cos^2(x) + \sin^2(x)$$

$$\text{Ⓑ} \quad \cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\text{Ⓐ}+\text{Ⓑ} = 1 + \cos(2x) = 2\cos^2(x)$$

$$\text{Ⓐ}-\text{Ⓑ} = 1 - \cos(2x) = 2\sin^2(x)$$

Rearranging:

Half-angle formulas

$$\boxed{\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}}$$

$$\int \cos^2(x)\sin^2(x)\,dx$$

$$= \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)dx$$

$$= \frac{1}{4}\int \left[1 - \cos^2(2x)\right]dx$$

Apply the half angle formula again

$$\cos^2(2x) = \frac{1+\cos(2\cdot(2x))}{2}$$

$$= \frac{1+\cos(4x)}{2}$$

$$= \frac{1}{4}\int \left[1 - \left\{\frac{1}{2} + \frac{1}{2}\cos(4x)\right\}\right] dx$$

$$= \frac{1}{4}\int \left[\frac{1}{2} - \frac{1}{2}\cos(4x)\right] dx$$

$$= \frac{1}{8}\int 1 - \cos(4x) \quad dx$$

$$= \frac{1}{8}\left[x - \frac{\sin(4x)}{4} + C_1\right]$$

$$= \boxed{\frac{1}{8}x - \frac{\sin(4x)}{32} + C_2}$$

constants of integration unknown

$C_2 = C_1/8$

m	n	method
odd >0	real #	split off a factor of sin(x) ↳ u-subst.
real #	odd >0	split off a factor of cos(x) ↳ u-subst.
even ≥ 0	even ≥ 0	use half-angle formulas

Ex: Challenge

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

m=1 is odd

$$= \int \sin^{1}(x)\cos^{-1}(x)\,dx$$

split off factor of sin(x)

$$= \int \cos^{-1}(x) \underbrace{\sin(x)dx}$$

$u = \cos(x) \qquad du = -\sin(x)\,dx$

$$= \int \frac{1}{u}(-du) = -\ln|u| + C$$

$$= -\ln|\cos(x)| + C$$

pull - into $\ln 1 \cdot 1$

$$= \ln|\cos^{-1}(x)| + C$$

$$= \boxed{\ln|\sec(x)| + C}$$

Ex: $\int \sin^{3/2}(x)\cos^5(x)\,dx$ $\qquad m = 3/2$, $n = 5$ odd

split off a factor of $\cos(x)$

$$= \int \sin^{3/2}(x) \underbrace{\cos^4(x)} \underbrace{\cos(x)\,dx}_{du}$$

$u = \sin(x) \qquad du = \cos(x)\,dx$

$$\cos^4(x) = [\cos^2(x)]^2 = [1 - \sin^2(x)]^2$$

$$= \int \sin^{3/2}(x)[1 - \sin^2(x)]^2 \cos(x)\,dx$$

$$= \int u^{3/2}[1 - u^2]^2\,du$$

$$= \int u^{3/2} \left(1 - u^2\right)^2 du$$

$$= \int u^{3/2} \left[1 - 2u^2 + u^4\right] du$$

$$= \int \left(u^{3/2} - 2u^{7/2} + u^{11/2}\right) du$$

$$= \frac{u^{5/2}}{5/2} - \frac{2u^{9/2}}{9/2} + \frac{u^{13/2}}{13/2} + C$$

$$= \boxed{\frac{2}{5}\sin^{5/2}(x) - \frac{4}{9}\sin^{9/2}(x) + \frac{2}{13}\sin^{13/2}(x) + C}$$