Lesson 12 - MA 166 LEC 200

8.3: Trigonometric Integrals - Part 2

Announcements:

- Exam 1 on Wed Feb 9 @ 6:30-7:30pm
- Review on Wed

GOALS:

Evaluate integrals with powers of tan/sec

WARM UP:

Use a trig identity to convert the following integral into one suitable for integration by parts

$$\int [1 - \cos(2x)]^{3/2}\,dx$$

[✓ (a) $2^{3/2}\int \sin^3(x)\,dx$]

(b) $2^{3/2}\int \cos^3(x)\,dx$

(c) $\int \sin^{3/2}(2x)\,dx$

Half angle formula

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$2\sin^2(x) = 1 - \cos(2x)$$

$$\int [2\sin^2(x)]^{3/2}\,dx$$

$$= 2^{3/2}\int \sin^3(x)\,dx$$

EXAM 1 INFO:

Wed Feb 9 @ 6:30pm - 7:30pm in ELLT

- ✱ Be in your seat by 6:15pm
- ✱ Check out seating chart

TA: Nikhil Mehra

Bring:

- #2 pencil
- PUID
- Recitation section number
- TA's name

Format:

- multiple choice
- 12 Questions (each 8 pts)

$$12 \times 8 + 4 = 100$$

I. Powers of tan(x) and sec(x)

Review: Last class

$$\int \sin^m(x)\cos^n(x)\,dx$$

$$\int \sin^m(x)\cos^n(x)\,dx$$

m	n	method
odd >0	real \#	split off a $\sin(x)$ ↳ u-substitution
real \#	odd >0	split off a $\cos(x)$ ↳ u-substition
even ≥ 0	even ≥ 0	Half-angle formulas

Today: Want to Solve

$$\int \tan^m(x)\sec^n(x)\,dx$$

Tools: u-substitution

- $u=\tan(x)$ $\quad du=\sec^2(x)\,dx$
- $u=\sec(x)$ $\quad du=\sec(x)\tan(x)\,dx$

Trig formula:

$$\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}=\frac{1}{\cos^2(x)}$$

$$1+\tan^2(x)=\sec^2(x)$$

Ex: Evaluate $\int \tan^3(x)\sec^{\textcircled{4}}(x)\,dx$

n even ≥ 0 → split off $\sec^2(x)$ → $du=\sec^2(x)dx$

$$\int \tan^3(x)\underbrace{\sec^2(x)}_{\text{convert to fcn of }\tan(x)}\underbrace{\sec^2(x)dx}_{du}$$

$u=\tan(x)$

$\sec^2(x)=1+\tan^2(x)$

$$=\int \underbrace{\tan^3(x)}_{u^3}\left[1+\underbrace{\tan^2(x)}_{u^2}\right]\underbrace{\sec^2(x)\,dx}_{du}$$

u^2

$$= \int u^3\left[1+u^2\right]du = \int u^3+u^5\,du$$

$$= \frac{u^4}{4}+\frac{u^6}{6}+C$$

$$= \boxed{\frac{\tan^4(x)}{4}+\frac{\tan^6(x)}{6}+C}$$

m	n	method
>0	even >0	split off $\sec^2(x)$ ↳ u substitution $u=\tan(x)$

Ex: Same example – different method

$$\int \tan^{③}(x)\sec^{④}(x)\,dx$$

>0

m is odd

split off $\underbrace{\tan(x)\sec(x)\,dx}_{du}$

$u=\sec(x)$

$$\int \underbrace{\tan^2(x)}_{}\ \underbrace{\sec^3(x)}_{u^3}\ \underbrace{\tan(x)\sec(x)\,dx}_{du}$$

write as a fn of $\sec(x)$ $\qquad \tan^2(x)=\sec^2(x)-1$

$$\int\left[\underbrace{\sec^2(x)}_{u^2}-1\right]\underbrace{\sec^3(x)}_{u^3}\ \underbrace{\tan(x)\sec(x)\,dx}_{du}$$

$$= \int\left[u^2-1\right]u^3\,du = \int u^5-u^3\,du$$

$$= \frac{u^6}{6}-\frac{u^4}{4}+C$$

$$= \boxed{\frac{\sec^6(x)}{6}-\frac{\sec^4(x)}{4}+C}$$

$$= \boxed{\frac{\sec^6(x)}{6} - \frac{\sec^7(x)}{4} + C}$$

Now we have 2 diff. solutions

$$\int \tan^3(x)\sec^4(x)\,dx$$

Q: Are these equivalent?

① $\frac{\tan^4(x)}{4} + \frac{\tan^6(x)}{6} + C$

② $\frac{\sec^6(x)}{6} - \frac{\sec^4(x)}{4} + C$

$\sec^2(x) = 1 + \tan^2(x)$

$$= \frac{[1+\tan^2(x)]^3}{6} - \frac{[1+\tan^2(x)]^2}{4} + C$$

$$= \frac{\tan^6 + 3\tan^4 + \cancel{3\tan^2} + 1}{6} - \left\{\frac{1 + \cancel{2\tan^2} + \tan^4}{4}\right\} + C$$

$$= \frac{\tan^6}{6} + \left(\frac{1}{2}\tan^4 - \frac{1}{4}\tan^4\right) + \frac{1}{6} - \frac{1}{4} + C$$

$$= \frac{\tan^6}{6} + \frac{\tan^4}{4} + \underbrace{\left(\frac{1}{6} - \frac{1}{4} + C\right)}_{C}$$

Two solutions are equivalent

m	n	method
>0	even >0	split off $\sec^2(x)$ u-subst
odd >0	>0	split off $\tan(x)\sec(x)$ u-subst.

even odd ?

Ex: $\int \tan^2(x) \sec(x)\,dx$

$$= \int \underbrace{\tan(x)}_{u} \cdot \underbrace{\tan(x)\sec(x)\,dx}_{dv} = \int u\,dv$$

Integration by parts

$u = \tan(x)$

$du = \sec^2(x)dx$

$dv = \tan(x)\sec(x)dx$

$v = \int dv = \int \cdots$

$= \sec(x)$

$$= u \cdot v - \int v\,du$$

$$= \tan(x) \cdot \sec(x) - \int \sec(x) \cdot \underbrace{\sec^2(x)}_{1+\tan^2(x)} dx$$

$$= \tan(x)\sec(x) - \int \sec^3(x)dx$$

$$= \tan(x)\sec(x) - \int (1+\tan^2(x))\sec(x)dx$$

$$\int \tan^2(x)\sec(x)dx = \tan(x)\sec(x) - \int \sec(x)dx - \int \tan^2(x)\sec(x)dx$$

$$2\int \tan^2(x)\sec(x)dx = \tan(x)\sec(x) - \int \sec(x)dx$$

$$= \tan(x)\sec(x) - \ln|\sec(x)+\tan(x)| + C$$

$$\int \sec(x)dx = \int \sec(x) \frac{(\sec(x)+\tan(x))}{(\sec(x)+\tan(x))} dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{} dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\, dx$$

$u = \sec(x) + \tan(x)$ $\qquad du = \sec(x)\tan(x) + \sec^2(x)\, dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sec(x) + \tan(x)| + C$$

$$2\int \tan^2(x)\sec(x)\,dx = \underbrace{\tan(x)\sec(x) - \ln|\sec(x)+\tan(x)|}_{} + C$$

$$\int \tan^2(x)\sec(x)\,dx = \frac{1}{2}\left[\qquad\qquad\right] + C$$

m	n	method
>0	even	split off $\sec^2(x)$
odd	>0	split of $\tan(x)\sec(x)$
even >0	odd $n>0$	Integration by Parts

Ex: $\int \tan^3(x)\,dx$

$= \int \tan(x)\tan^2(x)\,dx$ (reduction of order)

$= \int \tan(x)\left[\sec^2(x) - 1\right] dx$

degree dropped

$$= \int \tan(x)\sec^{\textcircled{2}}(x)\,dx - \int \tan(x)\,dx$$

$n=2 \longrightarrow$ split off $\sec^2(x)$

$u=\tan(x) \qquad du = \sec^2(x)\,dx$

$$= \int u\,du - \left[\text{review last class notes}\right]$$

$$= \frac{u^2}{2} + C - \ln\left|(\cos(x))^{-1}\right| + C$$

$$= \frac{\tan^2(x)}{2} - \ln|\sec(x)| + C$$

HW: $\displaystyle\int \tan^n(x)\,dx = \ldots$