8.4: Trig Substitutions

Announcements:
Exam 1 grades may not be posted until tonight

GOALS: Evaluate Integrals Using Trig Substitutions

Warm Up:

Which of the following u-substitutions will work to evaluate $\int \tan^5(x)\sec^4(x)\,dx$

✓ (a) $u = \tan(x)$

✓ (b) $u = \sec(x)$

$\boxed{\text{(c) both a and b}}$

(d) neither a nor b

$u = \tan(x)$
$du = \sec^2(x)\,dx$

$$\int u^5 \sec^2(x)\,du$$
$$\int u^5 \left[1 + \tan^2(x)\right] du$$
$$= \int u^5 (1 + u^2)\,du$$

$u = \sec(x)$
$du = \tan(x)\sec(x)\,dx$

$$\int \tan^4(x) \underbrace{\sec^3(x)}_{u^3} \underbrace{\tan(x)\sec(x)\,dx}_{du}$$
$$= \int \left[\sec^2(x) - 1\right]^2 \sec^3(x)\tan(x)\sec(x)\,dx$$
$$= \int \left[u^2 - 1\right]^2 u^3\,du$$

May give different solns $\longrightarrow$ solutions will be equivalent

using: $\tan^2(x) + 1 = \sec^2(x)$

Exam 1:

- Scores posted by tonight
- survey
- post a Benchmark
  if you got XX $\longrightarrow$ grade in class
  post on Wed

I Trig Substitutions.

I. Trig Substitutions:

GOAL: Remove " $\sqrt{}$ " in integration

3 types

Integral	Substitution	Simplified
$\sqrt{a^2-x^2}$		
$\sqrt{a^2+x^2}$		
$\sqrt{x^2-a^2}$		

Case: $\sqrt{a^2-x^2}$

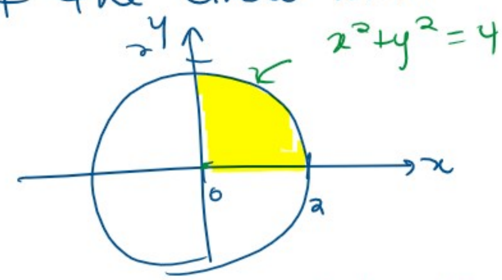
choose substitution $x = a\sin\theta$

then $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta}$

$= \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta}$

$= a\cos\theta$

Ex: Find the area of the circle with radius 2



$A = 4 \cdot$ 

$= 4\int_0^2 \sqrt{4-x^2}\,dx$

Limits of integration

@ $x=0 \rightarrow 2\sin\theta = 0$

$\theta = 0$

@ $x=2 \rightarrow 2\sin\theta = 2$

$\theta = \frac{\pi}{2}$

Here $a^2 = 4 \rightarrow a = 2$

Trig subst: $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$A = 4\int_0^{\frac{\pi}{2}} \sqrt{4-(2\sin\theta)^2} \cdot 2\cdot\cos\theta\, d\theta$

trig subst.

$$= 4\int_0^{\frac{\pi}{2}} 2\cos\theta \cdot 2\cos\theta \, d\theta$$

Trig Integral

→ Half angle formula

$$= 16\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= 16\int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2} \, d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} 1+\cos(2\theta)\,d\theta = 8\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 8\left[\frac{\pi}{2} + \frac{1}{2}\cancelto{0}{\sin\left(2\cdot\frac{\pi}{2}\right)} - \left\{0 + \frac{1}{2}\cancelto{0}{\sin(2\cdot 0)}\right\}\right]$$

$$= \frac{8\pi}{2} = \boxed{4\pi} = \pi r^2 \qquad r = 2$$

integral	substitution	simplified
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$\sqrt{a^2(1-\sin^2\theta)}$ $= a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$\sqrt{a^2+(a\tan\theta)^2}$ $= \sqrt{a^2(1+\tan^2\theta)}$ $= \sqrt{a^2\sec^2\theta}$ $= a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sqrt{(a\sec\theta)^2 - a^2}$ $= \sqrt{a^2(\sec^2\theta - 1)}$ $= \sqrt{a^2\tan^2\theta}$ $= a\tan\theta$

Ex: Find the arc length of parabola $y = x^2$ on $[0, 2]$

$$\int_0^2 \sqrt{1+\left\{\frac{dy}{dx}\right\}^2}\,dx \qquad \frac{dy}{dx} = 2x$$

$$L = \int_0^2 \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2}\, dx \qquad \frac{dy}{dx} = 2x$$

$$= \int_0^2 \sqrt{1 + (2x)^2}\, dx$$

$$= \int_0^2 \sqrt{1 + 4x^2}\, dx = 2\int_0^2 \sqrt{\left(\tfrac{1}{2}\right)^2 + x^2}\, dx$$

Here $a = \frac{1}{2}$ Trig subst: $x = \frac{1}{2}\tan\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

$$L = 2\int \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\tan\theta\right)^2} \cdot \tfrac{1}{2}\sec^2\theta\, d\theta$$

Ignore the limits of integration for now

$$= \cancel{2}\int \frac{1}{2}\sec\theta \cdot \frac{1}{\cancel{2}}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sec^3\theta\, d\theta$$

→ Trig Integral $m = 0 \quad n = 3$

→ Integration by Parts

IBP: $\int u\,dv = u\cdot v - \int v\,du$

$$= \frac{1}{2}\int \underbrace{\sec\theta}_{u = \sec\theta} \cdot \underbrace{\sec^2\theta\, d\theta}_{dv = \sec^2\theta\, d\theta}$$

$du = \tan\theta\sec\theta\, d\theta \qquad v = \int dv = \tan\theta$

$$= \frac{1}{2}\left[u\cdot v - \int v\,du\right]$$

$\tan^2\theta = \sec^2\theta - 1$

$$= \frac{1}{2}\left[\sec\theta\tan\theta - \int \tan\theta\cdot\tan\theta\sec\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta - \int [\sec^2\theta - 1]\sec\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta + \int\sec\theta\, d\theta - \int\sec^3\theta\, d\theta\right]$$

$\frac{1}{2}\int\sec^3\theta\, d\theta$

$$\frac{1}{2}\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta - \frac{1}{2}\int \sec^3\theta\, d\theta$$

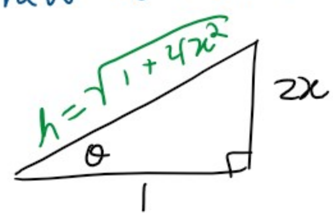
$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$L = \frac{1}{2}\int_a^b \sec^3\theta\, d\theta = \frac{1}{4}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_a^b$$

<u>WANT</u>: to evaluate the integral at limits $x=0$ and $x=2$

<u>SOLN</u>: Draw a reference triangle



$$x = \frac{1}{2}\tan\theta$$

$$\tan\theta = 2x$$

$$h^2 = 1^2 + (2x)^2$$

$$h = \sqrt{1+4x^2}$$

$$\sec\theta = \frac{h}{1} = \sqrt{1+4x^2}$$

$$L = \frac{1}{4}\left[\sqrt{1+4x^2}\cdot 2x + \ln\left|\sqrt{1+4x^2} + 2x\right|\right]\Bigg|_{x=0}^{x=2}$$

$$= \frac{1}{4}\Big[\sqrt{1+4\cdot 2^2}\cdot 2\cdot 2 + \ln\left|\sqrt{1\cdot 4\cdot 2^2} + 2\cdot 2\right|$$

$$- \cancelto{0}{\sqrt{1+0}\cdot 2\cdot 0} - \ln\left|\cancelto{0}{\sqrt{1+0}} + 2\cdot 0\right|\Big]$$

$$= \frac{1}{4}\left[4\sqrt{17} + \ln|4+\sqrt{17}|\right] = \boxed{\sqrt{17} + \frac{\ln|4+\sqrt{17}|}{4}}$$

<u>Ex:</u> $\int \frac{\sqrt{x^2-25}}{x}\,dx$

Trig subst: $x = a\sec\theta$

$a = 5$