Announcements:
Exam 1 grades may not be posted until tonight

8.4: Trig Substitutions - Part 1

GOALS: Evaluate Integrals Using Trig Substitutions

Warm Up:

Which of the following u-substitutions will work to evaluate $\int \tan^5(x)\sec^4(x)\,dx$

✓(a) $u = \tan(x)$

✓(b) $u = \sec(x)$

$\rightarrow$ $\boxed{\text{(c) both a and b}}$

(d) neither a nor b

$u = \tan(x)$
$du = \sec^2(x)$

$$\int \underbrace{\tan^5(x)}_{u^5} \sec^2(x) \cdot \underbrace{\sec^2(x)\,dx}_{du}$$

$$\int \tan^5(x) \cdot [\tan^2(x) + 1]\sec^2(x)\,dx$$

$$= \int u^5[u^2+1]\,du$$

$u = \sec(x)$
$du = \tan(x)\sec(x)dx$

$\tan^2 = \sec^2(x) - 1$

$$\int \underbrace{\tan^4(x)} \underbrace{\sec^3(x)}_{u^3} \underbrace{\tan(x)\sec(x)\,dx}_{du}$$

$$\int [\sec^2(x)-1]^2 \sec^3(x) \underbrace{\tan(x)\sec(x)dx}$$

$$= \int [u^2-1]^2 u^5\,du$$

NOTE! These may have solutions that look different

But they will be equivalent

$$\tan^2(x) + 1 = \sec^2(x)$$

Exam: - grades posted by tonight

<u>Exam</u>: - grades posted by tonight
- survey
- Benchmark posted on Wed

If you got XX on exam 1 — think of it as grade ¥

I. Trig Substitions:

<u>GOAL</u>: Remove "$\sqrt{}$" in integrals

Three kinds

Integral	Substitution	Simplified
$\sqrt{a^2-x^2}$		
$\sqrt{a^2+x^2}$		
$\sqrt{x^2-a^2}$		

<u>case</u>: $\sqrt{a^2-x^2}$

Choose substution $x = a\sin\theta$

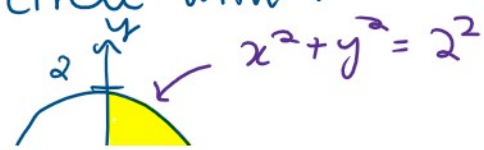
then $\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2}$

$= \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta}$

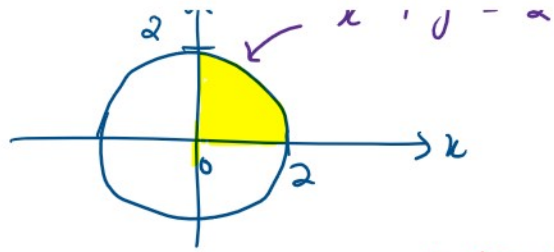
$= a\cos\theta$

<u>EX</u>: Find the area of the circle with radius 2

$A = 4 \times$  (know πr^2)



$A = 4 \times$ (know πr^2)



$$A = 4\int_0^2 \sqrt{4 - x^2}\,dx$$

Here $a = 2$ $\quad x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

@ $x = 0$ $\quad 2\sin\theta = 0$, $\theta = 0$

@ $x = 2$ $\quad 2\sin\theta = 2$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$

Trig Subst. $\rightarrow$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{2^2 - 2^2\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$\sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$= 4\int_0^{\frac{\pi}{2}} 2\cos\theta \cdot 2\cos\theta\, d\theta$$

Trig Integral
even power of $\cos\theta$
Half Angle Formula

$$\rightarrow = 16\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 16\int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\, d\theta = 8\int_0^{\frac{\pi}{2}} 1 + \cos(2\theta)\, d\theta$$

$$= 8\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 8\left[\frac{\pi}{2} + \frac{1}{2}\cancel{\sin\left(2\cdot\frac{\pi}{2}\right)}^{0} - 0 - \frac{1}{2}\cancel{\sin(2\cdot 0)}^{0}\right]$$

$$= \frac{8\pi}{2} = 4\pi = \pi r^2 \checkmark$$

Integral	Substitution	Simplified
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$\sqrt{a^2(1-\sin^2\theta)}$ $= \sqrt{a^2\cos^2\theta}$ $= a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$\sqrt{a^2 + a^2\tan^2\theta}$

$\sqrt{a^2+x^2}$ $\quad x = a\tan\theta$ $\quad \sqrt{a^2+a^2\tan^2\theta}$

$$= \sqrt{a^2(1+\tan^2\theta)}$$
$$= \sqrt{a^2\sec^2\theta}$$
$$= a\sec\theta$$

$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta$ $\quad \sqrt{a^2\sec^2\theta - a^2}$

$$= \sqrt{a^2(\sec^2\theta - 1)}$$
$$= \sqrt{a^2\tan^2\theta} = a\tan\theta$$

Ex: Find the arc length of the parabola $y = x^2$ on $[0, 2]$

$$L = \int_0^2 \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2}\, dx \qquad \frac{dy}{dx} = 2x$$

$$= \int_0^2 \sqrt{1+(2x)^2}\, dx = \int_0^2 2\sqrt{\frac{1+4x^2}{4}}\, dx$$

$$= 2\int_0^2 \sqrt{\left(\tfrac{1}{2}\right)^2 + x^2}\, dx$$

Here $a = \frac{1}{2}$

$x = \frac{1}{2}\tan\theta$

$$dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$L = 2\int \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2\tan^2\theta} \cdot \frac{1}{2}\sec^2\theta\, d\theta$$

$$= 2\int \sqrt{\left(\tfrac{1}{2}\right)^2\sec^2\theta} \cdot \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \cancel{2}\int \frac{1}{\cancel{2}} \cdot \sec\theta \cdot \frac{1}{2}\sec^2\theta\, d\theta$$

This Integral,

$$= \cancel{2} \int \frac{1}{\cancel{2}} \cdot \sec\theta \cdot \frac{1}{2} \sec\theta \, d\theta$$

$$L = \frac{1}{2} \int \sec^3\theta \, d\theta$$ ← Trig Integral Integration by Parts

IBP: $\int u \, dv = u \cdot v - \int v \, du$

$$L = \frac{1}{2} \int \sec\theta \cdot \sec^2\theta \, d\theta$$

$u = \sec\theta$ $\quad dv = \sec^2\theta \, d\theta$

$du = \tan\theta \sec\theta \, d\theta$ $\quad v = \int dv = \tan\theta$

$$= \frac{1}{2}\left[u \cdot v - \int v \, du \right]$$

$\tan^2\theta = \sec^2\theta - 1$

$$= \frac{1}{2}\left[\sec\theta \tan\theta - \int \tan\theta \cdot \tan\theta \sec\theta \, d\theta \right]$$

$$= \frac{1}{2}\left[\sec\theta \tan\theta + \int \sec\theta \, d\theta - \int \sec^3\theta \, d\theta \right]$$

$$L = \frac{1}{2} \int \sec^3\theta \, d\theta = \frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \int \sec\theta \, d\theta - \frac{1}{2} \int \sec^3\theta \, d\theta$$

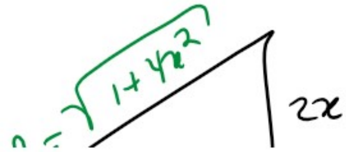
$$L = \frac{1}{2}\left[\int_a^b \sec^3\theta \, d\theta = \frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \int \sec\theta \, d\theta \right.$$
$$\left. = \frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \ln|\sec\theta + \tan\theta| + C \right]$$

$$L = \frac{1}{4}\left[\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| \right] \Big|_a^b$$

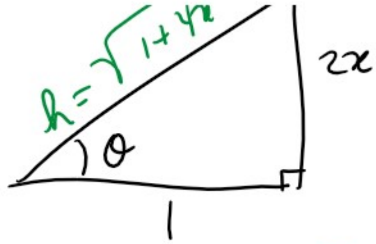
What are the limits of integration?

Draw a reference triangle ← Trig substitution

$$\boxed{x = \frac{1}{2} \tan\theta}$$



$\rightarrow \tan\theta = 2x = \frac{\sin\theta}{\cos\theta}$



$$\rightarrow \tan\theta = 2x = \frac{\sin\theta}{\cos\theta}$$

$$h^2 = 1^2 + (2x)^2$$

$$h = \sqrt{1+4x^2}$$

$$\sec\theta = \frac{h}{1} = \sqrt{1+4x^2}$$

$$L = \frac{1}{4}\left[\sqrt{1+4x^2}\cdot 2x + \ln\left|\sqrt{1+4x^2} + 2x\right|\right]_{x=0}^{x=2}$$

$$\boxed{L = \sqrt{17} + \frac{\ln\left|\sqrt{17}+4\right|}{4}}$$