Lesson 14 - MA 166 LEC 200

# 8.4: Trig Substitutions - Part 2

**Announcements:**

Academic Success Center Workshop
"Enhance Your Study Skills"
Feb 28 @ 5:30 pm

Link to Register

## GOALS:

- Evaluate Integrals Involving Trig Substitutions
- Complete the Square to Solve ∫

**WARM UP:** Use the Trig Subst. $x = 8\sec\theta$ to write

$$\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}}$$

as a Trig Integral

$$\int_a^b \frac{8\tan\theta\sec\theta\, d\theta}{\sqrt{8^2(\sec^2\theta-1)}} = \int_a^b \frac{8\tan\theta\sec\theta\, d\theta}{8\tan\theta}$$

What are the new limits of integration $a$ and $b$?

(A) $a = \frac{\pi}{2}$ $\quad b = \frac{\pi}{4}$

$\boxed{\text{(B) } a = \frac{\pi}{3} \quad b = \frac{\pi}{4}}$

$\boxed{b = \frac{\pi}{3}}$

(C) $a = \frac{\pi}{3}$ $\quad b = \frac{\pi}{2}$

$a$ : $\quad x = 8\sec\theta$

$8\sqrt{2} = 8\sec\theta$

$\sqrt{2} = \sec\theta$

$\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \rightarrow \theta = \frac{\pi}{4}$

$\boxed{a = \frac{\pi}{4}}$

## EXAM 1:

- Exam Booklets returned in Recitation Thurs Feb 24
- Benchmark Grade
- Grade Calculator

## I. Trig Substitutions:

| Integral | Substitution | Simplified |
|---|---|---|
| $\sqrt{a^2-x^2}$ | $x = a\sin\theta$ | $a\cos\theta$ |
| $\sqrt{a^2+x^2}$ | $x = a\tan\theta$ | $a\sec\theta$ |
| $\sqrt{x^2-a^2}$ | $x = a\sec\theta$ | $a\tan\theta$ |

Ex: $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

Here $a = 5$
$x = 5\sec\theta$
$dx = 5\tan\theta\sec\theta\,d\theta$

$$\int \frac{\sqrt{5^2\sec^2\theta - 5^2}}{\cancel{5}\cancel{\sec\theta}} \cdot \cancel{5}\tan\theta\,\cancel{\sec\theta}\,d\theta$$

$$= \int \sqrt{5^2\underbrace{(\sec^2\theta - 1)}_{\tan^2\theta}} \cdot \tan\theta\,d\theta = \int 5\tan\theta\cdot\tan\theta\,d\theta$$

$$= 5\int \tan^2\theta\,d\theta$$ ← Trig Integral → Trig Identity $\tan^2\theta = \sec^2\theta - 1$

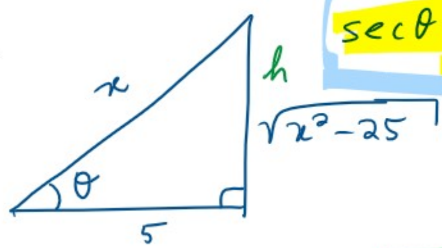
$$= 5\int[\sec^2\theta - 1]\,d\theta = 5[\tan\theta - \theta] + C$$

WANT: soln in terms of $x$

$x = 5\sec\theta$

$\boxed{\sec\theta = \frac{x}{5}}$

Draw a reference triangle




Pythag. $x^2 = 5^2 + h^2 \longrightarrow h = \sqrt{x^2-25}$

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

$$\sin\theta = \frac{\sqrt{x^2-25}}{x}$$

$$\boxed{\theta = \sec^{-1}\left(\frac{x}{5}\right)}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{x^2-25}}{x}\right)$$

$$\int = 5\left[\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\left(\frac{x}{5}\right)\right] + C$$

$$= \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$

## II. Completing the Square:

Evaluate $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$

First complete the square:

$$\cancel{x^2}+\cancel{4x}-5 = (x+2)^2 + b$$
$$= \cancel{x^2}+\cancel{4x}+4+b$$
$$b = -5-4 = -9$$

$$x^2+4x-5 = (x+2)^2 - 9 \qquad \sqrt{x^2-3^2}$$

$$\int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}\,dx$$

u-substitution

$u = x+2 \qquad du = dx$

$$\int_3^6 \frac{\sqrt{u^2-3^2}}{u}\,du$$

@ $x=1 \qquad u = 1+2 = 3$

@ $x=4 \qquad u = 4+2 = 6$

Trig substitution:

$a = 3$

$u = 3\sec\theta$

$du = 3\tan\theta\sec\theta\, d\theta$

$$\int_0^{\frac{\pi}{3}} \frac{\overbrace{\sqrt{(3\sec\theta)^2-3^2}}^{3\tan\theta}}{\cancel{3}\,\cancel{\sec\theta}} \cdot \cancel{3}\tan\theta\,\cancel{\sec\theta}\,d\theta$$

Limits:

@ $u=3$: $3 = 3\sec\theta$, $1 = \sec\theta$, $\cos\theta = 1$, $\theta = 0$

@ $u=6$: $6 = 3\sec\theta$, $2 = \sec\theta$, $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}} 3\tan\theta \cdot \tan\theta\,d\theta$$

$$= 3\int_0^{\frac{\pi}{3}} \left[\sec^2\theta - 1\right] d\theta$$

$$= 3\left[\tan\theta - \theta\right]_0^{\frac{\pi}{3}} = 3\left[\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3} - \left\{\overset{0}{\cancel{\tan 0}} - \overset{0}{\cancel{0}}\right\}\right]$$

$$= 3\left[\frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} - \frac{\pi}{3}\right] = 3\left[\frac{\sqrt{3}/\cancel{2}}{1/\cancel{2}} - \frac{\pi}{3}\right] = \boxed{3\sqrt{3} - \pi}$$

<u>Ex:</u> Evaluate $\displaystyle\int \frac{dx}{x^2-6x+34}$

Ex: Evaluate $\int \frac{}{x^2-6x+34}$

Complete the square to get a new integral

$$x^2-6x+34 = (x-3)^2 + b$$
$$= x^2-6x+9+b$$
$$b = 34-9 = 25$$

$$\int \frac{dx}{(x-3)^2+25}$$

u substitution
$u = x-3$ $du = dx$

$$\int \frac{du}{u^2+25}$$

Trig subst: $u = 5\tan\theta$
$du = 5\sec^2\theta\, d\theta$

$$\int \frac{5\sec^2\theta\, d\theta}{25\tan^2\theta+25} = \int \frac{\cancel{5}\,\cancel{\sec^2\theta}\, d\theta}{5^{\cancel{2}}(\cancel{\sec^2\theta})}$$

$$= \int \frac{d\theta}{5} = \frac{\theta}{5} + C$$

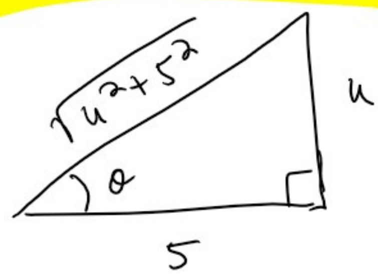
Want to convert back to $x$

Reference Triangle

SKIP

$u = 5\tan\theta$

$\tan\theta = \frac{u}{5}$




$\tan\theta = \frac{u}{5}$

$\theta = \tan^{-1}\left(\frac{u}{5}\right)$

$u = x-3$

soln: $\frac{\theta}{5} + C$

$\hookrightarrow \frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right) + C$

$\hookrightarrow \frac{1}{5}\tan^{-1}\left(\frac{x-3}{5}\right) + C$