Lesson 14 - MA 166 LEC 286

8.4: Trig Substitutions - Part 2

Announcements:

Academic Success Center Workshop
"Enhance Your Study Skills"
Feb 28 @ 5:30 pm

Link to Register

GOALS:

- Evaluate Integrals Involving Trig Substitutions
- Complete the Square to Solve

WARM UP: Use the Trig Subst. $x = 8\sec\theta$ to write

$$\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}}$$

as a Trig Integral

$$\int_a^b \frac{8\tan\theta\sec\theta\, d\theta}{\sqrt{8^2(\sec^2\theta-1)}} = \int_a^b \frac{8\tan\theta\sec\theta\, d\theta}{8\tan\theta}$$

What are the new limits of integration a and b?

(A) $a = \frac{\pi}{2}$ $\quad b = \frac{\pi}{4}$

$\boxed{\text{(B) } a = \frac{\pi}{4} \quad b = \frac{\pi}{3}}$

(C) $a = \frac{\pi}{3}$ $\quad b = \frac{\pi}{2}$

a:
$$x = 8\sec\theta$$
$$8\sqrt{2} = 8\sec\theta$$
$$\sqrt{2} = \sec\theta$$
$$\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \rightarrow \boxed{\theta = \frac{\pi}{4}}$$

Exam 1:

- Exam Booklets returned in Recitation on Thurs Feb 24
- Benchmark Grades

I. Trig Substitutions:

Integral	Substitution	Simplified
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a\cos\theta$

$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a\tan\theta$

Ex: $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

Here $a = 5$

$x = 5\sec\theta$

$dx = 5\tan\theta\sec\theta\,d\theta$

$$= \int \frac{\sqrt{5^2\sec^2\theta - 5^2}}{\cancel{5}\cancel{\sec\theta}} \cdot \cancel{5}\tan\theta\,\cancel{\sec\theta}\,d\theta$$

Trig Integral

$$= \int 5\tan\theta \cdot \tan\theta\,d\theta = 5\int \tan^2\theta\,d\theta$$

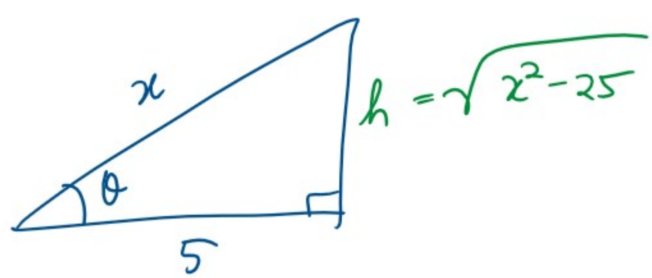
$$= 5\int (\sec^2\theta - 1)\,d\theta = 5[\tan\theta - \theta] + C$$

WANT - in terms of x

$x = 5\sec\theta \longrightarrow \sec\theta = \frac{x}{5}$

$\tan\theta = ?$

Draw a reference triangle



Pythag. $5^2 + h^2 = x^2$

$h = \sqrt{x^2-25}$

$\tan\theta = \frac{h}{5} = \frac{\sqrt{x^2-25}}{5}$

$\sec\theta = \frac{x}{5}$

$\theta = \sec^{-1}\left(\frac{x}{5}\right)$

$$\text{Integral} = 5\left[\tan\theta - \theta\right] + C$$

$$= 5\left[\frac{\sqrt{x^2-25}}{5} - \theta\right] + C$$

$$= \boxed{5\left[\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\left(\frac{x}{5}\right)\right] + C}$$

II. Completing the Square:

Evaluate $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$

First, complete the square

$$\cancel{x^2} + \cancel{4x} - 5 = (x+2)^2 + b$$

$$= \cancel{x^2} + \cancel{4x} + 4 + b$$

$$-9 = -5 - 4 = b$$

$$x^2 + 4x - 5 = (x+2)^2 - 9$$

$$\int_1^4 \frac{\sqrt{(x+2)^2 - 9}}{x+2}\,dx$$

To apply Trig Subst. $\sqrt{x^2 - 3^2}$

u-substitution

$u = x + 2$

$du = dx$

$$= \int_3^6 \frac{\sqrt{u^2 - 3^2}}{u}\,du$$

@$x=1$ $\quad u = x+2 = 1+2 = 3$

@$x=4$ $\quad u = x+2 = 4+2 = 6$

Trig substitution

$u = 3\sec\theta$

$du = 3\tan\theta\sec\theta\,d\theta$

@$u=3$ $\quad 3 = 3\sec\theta \rightarrow \cos\theta = 1$

$\theta = 0$

@$u=6$ $\quad 6 = 3\sec\theta \rightarrow \cos\theta = \frac{1}{2}$

$u = 3\sec\theta$ $\quad$ @ $u=3$ $\quad$ $3 = 3\sec\theta \rightarrow \cos\theta = 1$, $\theta = 0$

$du = 3\tan\theta\sec\theta\, d\theta$ $\quad$ @ $u=6$ $\quad$ $6 = 3\sec\theta \rightarrow \cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

$$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{3^2(\sec^2\theta - 1)}}{\cancel{3}\cancel{\sec\theta}} \cdot \cancel{3}\tan\theta\, \cancel{\sec\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} 3\tan\theta \cdot \tan\theta\, d\theta = 3\int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta$$

$$3\left[\tan\theta - \theta\right]_0^{\frac{\pi}{3}} = 3\left[\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3} - \left\{\cancel{\tan(0)}^{\,0} - 0\right\}\right]$$

$$= 3\left[\sqrt{3} - \frac{\pi}{3}\right] = \boxed{3\sqrt{3} - \pi}$$

Ex: Evaluate $\displaystyle\int \frac{dx}{x^2 - 6x + 34}$

Complete the Square

$$x^2 - 6x + 34 = (x-3)^2 + 25$$

$$= \int \frac{dx}{(x-3)^2 + 25}$$

u-substitution

$u = x - 3$ $\quad$ $du = dx$

$$= \int \frac{du}{u^2 + 25}$$

Trig substitution

$u = 5\tan\theta$

$du = 5\sec^2\theta\, d\theta$

$$= \int \frac{5\sec^2\theta\, d\theta}{25\tan^2\theta + 25} = \int \frac{\cancel{5}\cancel{\sec^2\theta}}{\cancel{25}\,\cancel{\sec^2\theta}}\, d\theta$$

$$= \int \frac{d\theta}{5} = \frac{\theta}{5} + C$$

Want soln in terms of x

$$= \int \frac{d\theta}{5} = \frac{\theta}{5} + C$$ Want soln of x

$u = 5\tan\theta \rightarrow \tan\theta = \frac{u}{5}$

Don't need a reference triangle

$\theta = \tan^{-1}\left(\frac{u}{5}\right)$

$u = x - 3$

$$= \frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right) + C$$

$$= \boxed{\frac{1}{5}\tan^{-1}\left(\frac{x-3}{5}\right) + C}$$

<u>Ex</u>: Evaluate $\int \frac{dx}{\sqrt{9+x^2}}$

Trig subst.

$x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

$$= \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9(\underbrace{1+\tan^2\theta}_{\sec^2\theta})}} = \int \frac{\cancel{3}\, \sec^{\cancel{2}}\theta \, d\theta}{\cancel{3}\, \cancel{\sec\theta}}$$

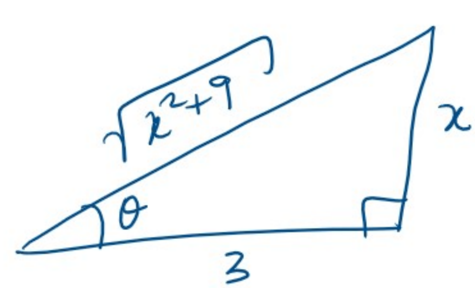
$$= \int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

WANT to convert back to x

$x = 3\tan\theta \rightarrow \tan\theta = \frac{x}{3}$

$\sec\theta = ?$

Reference Triangle



Reference Triangle

$$\sec\theta = \frac{\sqrt{x^2+9}}{3}$$

$$= \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$

$$= \ln\left(\frac{1}{3}\left|\sqrt{x^2+9} + x\right|\right) + C$$

$$= \ln\left(\frac{1}{3}\right) + \ln\left|\sqrt{x^2+9} + x\right| + C$$

both are constants
combine

$$= \boxed{\ln\left|\sqrt{x^2+9} + x\right| + C}$$