Lesson 16 - MA 166 LEC 200

8.5: Partial Fractions

Announcements:
Final Exam Scheduled
Monday May 2 @ 10:30am-12:30pm
in ELLT

GOALS:

- find a partial fraction decomposition
- Evaluate Integrals involving partial fractions

WARM-UP: Complete the Square to rearrange the integral

$$\int \frac{dx}{\sqrt{x^2-6x+13}}$$

(A) $\int \frac{dx}{\sqrt{(x-2)^2+9}}$ (B) $\int \frac{dx}{\sqrt{(x+3)^2+4}}$ (C) $\boxed{\int \frac{dx}{\sqrt{(x-3)^2+4}}}$

I. Partial Fractions:

Want to compute $\int \frac{P(x)}{Q(x)}\,dx$

where $P(x)$, $Q(x)$ are polynomials
degree of $P(x)$ < degree of $Q(x)$

→ Used in Laplace Transforms → MA 266, 303
→ circuits + control theory

Idea: $\frac{3}{x-3} + \frac{4}{x+2} = ?$

Find a common denominator: $(x-3)(x+2)$

$$\frac{3}{(x-3)} \cdot \frac{(x+2)}{(x+2)} + \frac{4}{x+2}\left(\frac{x-3}{x-3}\right) =$$

$$\frac{3x+6+4x-12}{(x-3)(x+2)} = \frac{7x-6}{(x-3)(x+2)}$$

← difficult

$(x-3)(x+2)$ $\qquad$ $(x-3)(x+2)$

$$\frac{3}{x-3} + \frac{4}{x+2} = \frac{7x-6}{x^2-x-6}$$

difficult to integrate

easy to integrate

TODAY

partial fraction decomposition

Process: Given a rational function

$$\frac{3x}{x^2+2x-8}$$

1. Factor the denominator: $\dfrac{3x}{(x+4)(x-2)}$

2. Expand this to be a sum of fractions

$$\frac{3x}{(x+4)(x-2)} = \frac{A}{x-2} + \frac{B}{x+4}$$

A, B are unknown constants

3. To solve for A, B, multiply both sides by $(x+4)(x-2)$

$$\frac{3x}{\cancel{(x+4)}\cancel{(x-2)}}\cancel{(x+4)}\cancel{(x-2)} = \frac{A}{\cancel{x-2}}(x+4)\cancel{(x-2)} + \frac{B}{\cancel{(x+4)}}\cancel{(x+4)}(x-2)$$

$$3x = A(x+4) + B(x-2)$$

4. Collect like terms

$$3x = [A+B]x + [4A-2B]$$

zero for all x $\rightarrow$ $$0 = [A+B-3]x + [4A-2B]$$

x is a variable

$\rightarrow$ $A + B - 3 = 0$
$4A - 2B = 0$ } set of 2 equations, 2 unknowns

Method of Elimination

$4A = 2B \rightarrow \boxed{B = 2A}$

$A + B - 3 = 0$

$A + 2A - 3 = 0 \rightarrow 3A = 3$

$\boxed{A = 1}$

$B = 2A = 2 \cdot 1$

$\boxed{B = 2}$

Integral $\int \frac{3x}{(x-2)(x+4)} \, dx = \int \frac{A}{x-2} + \frac{B}{x+4} \, dx$

$= \int \frac{1}{x-2} + \frac{2}{x+4} \, dx$

$= \ln|x-2| + 2\ln|x+4| + C$

$= \ln|x-2| + \ln|(x+4)^2| + C$

$= \boxed{\ln|(x-2)(x+4)^2| + C}$

Three cases:

type	equation	decomposition
simple linear	$\frac{P(x)}{(x-2)(x+4)}$	$\frac{A}{x-2} + \frac{B}{x+4}$
repeated linear	$\frac{P(x)}{(x-2)^3}$	$\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$

repeated linear	$\frac{P(x)}{(x-2)^3}$	$\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$
irreducible quadratic	$\frac{P(x)}{ax^2+bx+c}$	$\frac{Ax+B}{ax^2+bx+c}$

Ex: Find the partial fractions decomp.

$$\frac{5x^2-3x+2}{x^3-2x^2}$$

1. Factor denom.

$$\frac{5x^2-3x+2}{x^2(x-2)}$$

repeated linear → x^2 $(x-2)$ simple linear

2. Set up decomp.

$$\frac{5x^2-3x+2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

Ex: $$\frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}$$

simple linear → $(x-2)$ (x^2-2x+3) ← irreducible quadratic

1. Factor the denom. ✓

2. $$\frac{7x^2-13x+13}{(x-2)(x^2-2x+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}$$

Ex: $$\frac{3x^2+1}{x(x+1)^2(x^2+9)}$$

Find the partial fraction decomposition

$$3x^2+1 \quad = \quad A + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{x^2+9}$$

$$\frac{3x^2+1}{x(x+1)^2(x^2+9)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{x^2+9}$$

Ex: $\displaystyle\int \frac{3x^2+7x-2}{x(x+1)(x-2)}\,dx$

1. Already factored

2. PF Decomp.

$$\frac{3x^2+7x-2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

3. Multiply both sides by denom.

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

4. Solve for A, B, C

   1. collect like terms, set coeff = 0
      3 eqns and 3 unknowns A, B, C

   2. Evaluate at values of x
      Choose x to be roots of the common denominator

$$x(x+1)(x-2) = 0$$

roots: 0, −1, 2

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

@ $x=0$ $\quad 3\cdot\cancel{0^2}+\cancel{7\cdot 0}-2 = A(0+1)(0-2) + \cancel{B\cdot 0\cdot(0-2)}$

@ $x=0$ $\qquad \cancel{3\cdot 0^2}+\cancel{7\cdot 0}-2 = A(0+1)(0-2) + \cancel{B\cdot 0\cdot(0-2)} + \cancel{C\cdot 0\cdot(0+1)}$

$$-2 = A(1)(-2) = -2A \qquad \boxed{A=+1}$$

@ $x=-1$ $\qquad 3(-1)^2+7(-1)-2 = A\underbrace{(-1+1)}_{\to 0}(-1-2) + B(-1)(-1-2) + C(-1)\underbrace{(-1+1)}_{\to 0}$

$$3-7-2 = -6 = B(-1)(-3) = 3B \qquad \boxed{B=-2}$$

@ $x=2$ $\qquad 24 = 6C \longrightarrow \boxed{C=4}$

$$\int \frac{3x^2+7x-2}{x(x+1)(x-2)}\,dx = \int\left[\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}\right]dx$$

$$= \int\left[\frac{1}{x}+\frac{-2}{x+1}+\frac{4}{x-2}\right]dx$$

$$= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + C$$

$$= \ln|x| + \ln|(x+1)^{-2}| + \ln|(x-2)^4| + C$$

$$= \ln\left(\frac{|x|(x-2)^4}{(x+1)^2}\right) + C$$