8.5: Partial Fractions

GOALS:

- find a partial fraction decomposition
- Evaluate Integrals involving partial fractions

Announcements:
Final Exam Scheduled
Monday May 2 @ 10:30am-12:30pm
in ELLT

WARM-UP: Complete the Square to rearrange the integral

$$\int \frac{dx}{\sqrt{x^2-6x+13}}$$

(A) $\int \frac{dx}{\sqrt{(x-2)^2+9}}$ (B) $\int \frac{dx}{\sqrt{(x+3)^2+4}}$ $\boxed{(C) \int \frac{dx}{\sqrt{(x-3)^2+4}}}$

I. Partial Fractions:

Want $\int \frac{P(x)}{Q(x)}\,dx$ where $P(x), Q(x)$ are polynomials and degree $P(x) <$ degree $Q(x)$

Laplace Transforms — MA 266, 303
— used in circuits, control theory

Idea: $\frac{3}{x-3} + \frac{4}{x+2} = \ ?$

find common denominator

$$\frac{3}{x-3}\left(\frac{x+2}{x+2}\right) + \frac{4}{x+2}\left(\frac{x-3}{x-3}\right) = \frac{3x+6+4x-12}{(x-3)(x+2)}$$

$$\frac{3}{x-3} + \frac{4}{x+2} = \frac{7x-6}{x^2-x-6}$$

easy to integrate

← difficult to integrate

TODAY

partial fraction decomposition

rational function $\frac{P(x)}{Q(x)}$

PROCESS: Given a rational function $\frac{P(x)}{Q(x)}$

$$\frac{3x}{x^2+2x-8}$$

1. Factor the denominator $\quad \frac{3x}{(x+4)(x-2)}$

2. Expand to be a sum of fractions

$$\frac{3x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}$$

A, B unknown constants

3. multiply both sides by common denominator

$$\frac{3x}{\cancel{(x-2)}\cancel{(x+4)}}\cancel{(x-2)}\cancel{(x+4)} = \frac{A}{\cancel{x-2}}\cancel{(x-2)}(x+4) + \frac{B}{\cancel{x+4}}(x-2)\cancel{(x+4)}$$

$$3x = A(x+4) + B(x-2)$$

4. Solve for A and B

Collect like terms

$$3x = [A+B]x + [4A-2B]$$

always zero $\rightarrow$ $0 = [A+B-3]x + [4A-2B]$

variable

$$\left.\begin{aligned} A+B-3 &= 0 \\ 4A-2B &= 0 \end{aligned}\right\}$$ set of 2 eqns, 2 unknowns A, B

Method of Elimination

$4A = 2B \longrightarrow B = 2A$

$\boxed{B=2}$

$A + B - 3 = 0$

$A + 2A = 3 \qquad 3A = 3 \qquad \boxed{A=1}$

$$\frac{3x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4} = \frac{1}{x-2} + \frac{2}{x+4}$$

$$\int \frac{3x}{(x-2)(x+4)}\,dx = \int \left[\frac{1}{x-2} + \frac{2}{x+4} \right] dx$$

$$= \ln|x-2| + 2\ln|x+4| + C$$

$$= \ln|x-2| + \ln|(x+4)^2| + C$$

$$= \boxed{\ln|(x-2)(x+4)^2| + C}$$

3 Types of PF.

Type	Equation	Decomposition
simple linear	$\frac{P(x)}{(x-2)(x+4)}$	$\frac{A}{x-2} + \frac{B}{x+4}$
repeated linear	$\frac{P(x)}{(x-2)^3}$	$\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$
irreducible quadratic	$\frac{P(x)}{x^2+x-1}$	$\frac{Ax+B}{x^2+x-1}$

Ex: Find the partial fraction decompostion of

$$\frac{5x^2-3x+2}{x^3-2x^2}$$

1. Factor the denom: $\frac{5x^2-3x+2}{x^2(x-2)}$

repeated linear

simple linear

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

Ex: $\dfrac{7x^2-13x+13}{(x-2)(x^2-2x+3)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2-2x+3}$

simple linear

irreducible quadratic

Ex: $\dfrac{3x^2+1}{x(x+1)^2(x^2+9)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} + \dfrac{Dx+E}{x^2+9}$

Ex: $\displaystyle\int \frac{3x^2+7x-2}{x(x+1)(x-2)}\,dx$

1. Factor ✓
2. Decomp $\dfrac{3x^2+7x-2}{x(x+1)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-2}$
3. Multiply both sides by $x(x+1)(x-2)$

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

4. Solve for A, B, C
   1. Collected like terms, set coeff $=0$
      3 eqns, 3 unknowns A, B, C
   2. Evaluate at roots of the common denom
      $x(x+1)(x-2) = 0$
      $x = 0, -1, +2$

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

@ $x=0$ $\quad \cancel{3\cdot 0^2} + \cancel{7\cdot 0} - 2 = A(0+1)(0-2) + \cancel{B\cdot 0\cdot(0-2)}^{\,0} + \cancel{C\cdot 0\,(0+1)}^{\,0}$

$$-2 = A\cdot(1)(-2) = -2A \rightarrow \boxed{A=1}$$

$$@x=2 \qquad 3\cdot 2^2 + 7\cdot 2 - 2 = A(2+1)\cancel{(2-2)}^{\,0} + B\cdot\cancel{2(2-2)}^{\,0} + C\cdot 2(2+1)$$

$$24 = 12 + 14 - 2 = 6C \qquad \boxed{C=4}$$

$$@x=-1 \qquad -6 = 3B \rightarrow \boxed{B=-2}$$

$$\int \frac{3x^2+7x-2}{x(x+1)(x-2)}\,dx = \int \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}\right) dx$$

$$= \int \left[\frac{1}{x} + \frac{-2}{x+1} + \frac{4}{x-2}\right] dx$$

$$= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + C$$

$$= \ln|x| + \ln|(x+1)^{-2}| + \ln|(x-2)^4| + C$$

$$= \boxed{\ln\left|\frac{x(x-2)^4}{(x+1)^2}\right| + C}$$