Lesson 17 - MA 166 LEC 200

8.5: Partial Fractions - Part 2

Announcements:

Final Exam - Mon May 2 @ 10:30am
↳ Religious Holiday → email kthood@purdue. to request alternate time

Exam 2 → Wed Mar 9 @ 6:30pm
• study guide will be posted monday

WARM UP:

Fall 2015 Exam 2 Question # 2

2. The partial fraction decomposition of $\frac{1}{x^3+x}$ is

A. $\cancel{\frac{1}{x^3} - \frac{1}{x}}$

B. $\cancel{\frac{1}{x^3} + \frac{1}{2x^2} - \frac{1}{2x}}$

C. $\boxed{\frac{1}{x} - \frac{x}{1+x^2}}$

D. $\frac{2}{1+x^2} - \frac{3}{x}$

E. $\cancel{\frac{1}{x} - \frac{2}{x^2} + \frac{3}{1+x^2}}$

$$\frac{1}{\underset{\text{simple linear}}{x}\,\underset{\text{quadratic}}{(x^2+1)}} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Solve for A, B, C

★ Exam 1 - Grade Correction - Q#12

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12. The work needed to stretch a spring from its equilibrium length of 1 m to a length of 2 m is 6 J. How much work is needed to stretch the spring from a length of 2.5 m to a length of 3.5 m ?

$\boxed{\text{A. 24 J}}$

B. 3 J

~~C.~~ 12 J

D. 6 J

E. 18 J

$$6 = W = \int_0^1 kx\,dx = \left[\frac{kx^2}{2}\right]_0^1 = \frac{k}{2}$$

$$k = 12$$

$$W = \int_{1.5}^{2.5} 12x\,dx = \left[\frac{12x^2}{2}\right]_{\frac{3}{2}}^{\frac{5}{2}}$$

$$= 6\left[\left(\frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] = 6\left[\frac{25-9}{4}\right]$$

$$= 6\left(\frac{16}{4}\right) = 6 \cdot 4 = \boxed{24}$$

Regrade Request Process

I. Partial Fractions:

Type	Eqn	Decomposition
simple linear	$\frac{P(x)}{(x-1)(x-2)(x-3)}$	$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$
repeated	$P(x)$	$A + B$

repeated linear	$\frac{P(x)}{(x+1)^2}$	$\frac{A}{x+1} + \frac{B}{(x+1)^2}$
simple irreducible quadratic	$\frac{P(x)}{(x^2+1)}$	$\frac{Ax+B}{x^2+1}$
repeated irreducible quadratic	$\frac{P(x)}{[x^2+1]^2}$	$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{[x^2+1]^2}$

Ex: $\displaystyle\int \frac{7x^2-13x+13}{\underbrace{(x-2)}_{SL}\underbrace{(x^2-2x+3)}_{SQ}}\,dx$

1. Factor denominator ✓

2. $\dfrac{7x^2-13x+13}{(x-2)(x^2-2x+3)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2-2x+3}$

3. Multiply by the common denominator

$$7x^2-13x+13 = A(x^2-2x+3) + (Bx+C)(x-2)$$

4. Solve for A, B, C

Want roots of common denom.

$(x-2)(x^2-2x+3) = 0$

$\boxed{x=2}$

complete the square

$(x-1)^2+2 = 0$

$\sqrt{(x-1)^2} = \sqrt{-2}$ ← complex valued

$x = 1 \pm \sqrt{2}\,i$

Use easy to evaluate values

$x=0, \quad x=1$

$$7x^2-13x+13 = A(x^2-2x+3) + (Bx+C)(x-2)$$

$\boxed{@x=2}$ $\quad 7\cdot 2^2 - 13\cdot 2 + 13 = A(2^2-2\cdot 2+3)$
$+ \cancel{(B\cdot 2+C)(2-2)} \to 0$

$\boxed{@x=2}$ $\quad 7\cdot 2^2 - 13\cdot 2 + 13 = A(2 - 2\cdot 2 + 3) + \cancel{(B\cdot 2 + C)(2-2)} \rightarrow 0$

$15 = 28 - 26 + 13 = 3A \qquad \boxed{A = 5}$

$\boxed{@x=0}$ $\quad 7\cdot \cancel{0^2} - \cancel{13\cdot 0} + 13 = 5(\cancel{0^2} - \cancel{2\cdot 0} + 3) + (\cancel{B\cdot 0} + C)(0-2)$

$13 = 15 - 2C$

$-2 = -2C \qquad \boxed{C = 1}$

$\boxed{@x=1}$ $\quad 7 = 10 - B - 1 \rightarrow \boxed{B = 2}$

Plug into integral

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx = \int \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}\,dx$$

$$= \int \frac{5}{x-2} + \frac{2x+1-2+2}{x^2-2x+3}\,dx$$

$u = x^2 - 2x + 3$

$du = (2x-2)\,dx$

$= \ln|\cancel{x^2-2x+3}|$

$$= \int \frac{5}{x-2}\,dx + \underbrace{\int \frac{2x-2}{x^2-2x+3}\,dx}_{=\ln|x^2-2x+3|} + \boxed{\int \frac{3}{x^2-2x+3}\,dx}$$

complete the square

Trig Subst.

$$= 5\ln|x-2| + \int \frac{du}{u} + 3\int \frac{1}{(x-1)^2+2}\,dx$$

$v = x - 1$

$dv = dx$

$$= 5\ln|x-2| + \ln|u| + 3\int \frac{dv}{v^2+2}$$

$v = \sqrt{2}\tan\theta$

$dv = \sqrt{2}\sec^2\theta\,d\theta$

$$= 5\ln|x-2| + \ln|u| + 3\int \frac{\sqrt{2}\,\cancel{\sec^2\theta}\,d\theta}{(\sqrt{2}\tan\theta)^2 + 2} \quad = 2\cancel{\sec^2\theta}$$

$$= 5\ln|x-2| + \ln|u| + \frac{3}{\sqrt{2}}\int d\theta$$

$= 5\ln|x-2| + \ln|u| + \frac{3}{\sqrt{2}}\,\theta + C$

$u = x^2 - 2x + 3$

$v = \sqrt{2}\tan\theta \longrightarrow \theta = \tan^{-1}\left(\frac{v}{\sqrt{2}}\right)$

$v = x - 1 \qquad \theta = \tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)$

$$\boxed{= 5\ln|x-2| + \ln|x^2-2x+3| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C}$$

II. Repeated Quadratic Factor

SL $\displaystyle\int \frac{2}{x(x^2+1)^2}\,dx \longleftarrow$ repeated quadratic

1. Factor ✓
2. Expand out

$$\frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

3. Multiply

$$2 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$

4. Solve for A, B, C, D, E

roots of common denominator

$x(x^2+1)^2 = 0 \longrightarrow x = 0, \pm i$

collect like terms instead

$$0 = [A+B]x^4 + [C+D]x^3 + [2A+B+E]x^2 + Cx + [A-2]$$

$A + B = 0 \leftarrow \boxed{B = -2}$

$\cancel{C}^{0} + D = 0 \rightarrow \boxed{D = 0}$

$\boxed{C = 0}$

$A - 2 = 0 \rightarrow \boxed{A = 2}$

$A+B=0$ $\quad$ $1=$ ...

$\cancel{C}^{0}+D=0 \rightarrow \boxed{D=0}$ $\quad$ $A-2=0 \rightarrow \boxed{A=2}$

$\rightarrow 2A+B+E=0$

$2\cdot(2)+(-2)+E=0 \rightarrow \boxed{E=-2}$

Integral

$$\int \frac{2}{x(x^2+1)^2}\,dx = \int \frac{A}{x} + \frac{Bx+\cancel{C}^{0}}{x^2+1} + \frac{\cancel{Dx}^{0}+E}{(x^2+1)^2}\,dx$$

$$= \int \frac{2}{x} + \underbrace{\int \frac{-2x}{x^2+1}}_{u\text{-subst.}} + \underbrace{\int \frac{-2}{(x^2+1)^2}\,dx}_{\text{trig substitution}}$$

$x=\tan\theta$

$dx=\sec^2\theta\,d\theta$

$$-2\int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2}$$

$$=-2\int \frac{\cancel{\sec^2\theta}}{\sec^{\cancel{4}}\theta}\,d\theta$$

$$=-2\int \frac{1}{\sec^2\theta}\,d\theta$$

$$=-2\int \cos^2\theta\,d\theta$$

$$=-2\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

There is a correction to the last example

The values of D and E are switched $\rightarrow$ easier integral!

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*Last Example:

$$\frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiply by $x(x^2+1)^2$

$$\frac{2\,\cancel{x(x^2+1)^2}}{\cancel{x(x^2+1)^2}} = \frac{A}{\cancel{x}}\cancel{x}(x^2+1)^2 + \frac{Bx+C}{\cancel{x^2+1}}\cdot x(\cancel{x^2+1})^2 + \frac{Dx+E}{\cancel{(x^2+1)^2}}\cdot x\cancel{(x^2+1)^2}$$

$$2 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$

$$= A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2+Ex$$

$$2 = A(x^4+2x^2+1) + Bx^4+Bx^2+Cx^3+Cx + Dx^2+Ex$$

Now collect like terms:

$$0 = \underbrace{[A+B]}_{=0}x^4 + \underbrace{Cx^3}_{=0} + \underbrace{[2A+B+D]}_{=0}x^2 + \underbrace{[C+E]}_{=0}x + \underbrace{[A-2]}_{=0}$$

$A+B=0$

$\boxed{C=0}$

$2A+B+D=0$

$\cancel{C}^{0}+E=0 \rightarrow \boxed{E=0}$

$A-2=0 \rightarrow \boxed{A=2}$

$2+B=0 \rightarrow \boxed{B=-2}$

$2A+B+D=0$

$2\cdot 2+(-2)+D=0$

$\boxed{D=-2}$

$$\int \frac{2}{x(x^2+1)^2}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\,dx$$

plug in $A=2$, $B=-2$, $C=0$, $D=-2$, $E=0$

$$= \int \frac{2}{x}\,dx + \int \frac{-2x}{x^2+1}\,dx + \int \frac{-2x}{(x^2+1)^2}\,dx$$

use u-substitution

this integral is much easier now

$u = x^2+1$

$du = 2x\,dx$

$$= \int -\frac{du}{u^2}$$

$$= \left[-\frac{u^{-1}}{-1}\right] = \frac{1}{u}$$

$$= \frac{1}{x^2+1}$$