Lesson 17 - MA 166 LEC 200

8.5: Partial Fractions - Part 2

Announcements:

Final Exam - Mon May 2 @ 10:30am
↳ Religious Holiday → email kthood@purdue.
to request alternate time

Exam 2 → Wed Mar 9 @ 6:30 pm
• study guide will be posted monday

WARM UP:

Fall 2015 Exam 2 Question # 2

2. The partial fraction decomposition of $\dfrac{1}{x^3+x}$ is

A. ~~$\dfrac{1}{x^3} - \dfrac{1}{x}$~~

B. ~~$\dfrac{1}{x^3} + \dfrac{1}{2x^2} - \dfrac{1}{2x}$~~

C. $\boxed{\dfrac{1}{x} - \dfrac{x}{1+x^2}}$

D. $\dfrac{2}{1+x^2} - \dfrac{3}{x}$

E. ~~$\dfrac{1}{x} - \dfrac{2}{x^2} + \dfrac{3}{1+x^2}$~~

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

simple linear

quadratic

$$1 = A(x^2+1) + (Bx+C)x$$

$$0 = \underbrace{[A+B]}_{=0}x^2 + \underbrace{C}_{=0}x + \underbrace{[A-1]}_{=0}$$

★ Exam 1 - Grade Correction - Q#12

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12. The ~~work~~ force needed to stretch a spring from its equilibrium length of 1 m to a length of 2 m is 6 J. How much work is needed to stretch the spring from a length of 2.5 m to a length of 3.5 m ?

A. $\boxed{\text{24 J}}$

B. 3 J

C. ~~12 J~~

D. 6 J

E. 18 J

$$6 = W = \int_0^1 kx\,dx = \left(\frac{k x^2}{2}\right)_0^1$$

$$6 = \frac{k}{2} \rightarrow \boxed{k = 12}$$

$$W = \int_{1.5}^{2.5} 12x\,dx = \left[\frac{12x^2}{2}\right]_{3/2}^{5/2}$$

$$= 6\left[\left(\frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] = 6\cdot\left[\frac{25-9}{4}\right]$$

$$= 6\cdot\frac{16}{4} = \boxed{24}$$

Regrade Request

~~Key~~

$$= 6 \cdot \frac{16}{4} = \boxed{24}$$

I. Partial Fractions:

Type	Eqn	Decomposition
simple linear	$\frac{P(x)}{(x-1)(x-2)(x-3)}$	$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$
repeated linear	$\frac{P(x)}{(x-1)^2}$	$\frac{A}{x-1} + \frac{B}{(x-1)^2}$
simple irreducible quadric	$\frac{P(x)}{(x^2+1)}$	$\frac{Ax+B}{x^2+1}$
repeated irreducible quadratic	$\frac{P(x)}{(x^2+1)^2}$	$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

$$\int \frac{7x^2 - 13x + 13}{\underbrace{(x-2)}_{SL}\underbrace{(x^2-2x+3)}_{SQ}}\, dx$$

1. Factor the denom. ✓

2. Expand out

$$\frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}$$

3. Multiply both sides by common denom

$$7x^2 - 13x + 13 = A(x^2-2x+3) + (Bx-C)(x-2)$$

4. Solve for A, B, C

evaluate values of x

roots of common denom

$$(x-2)(x^2-2x+3) = 0$$

$$\boxed{x=2}$$

$$x^2-2x+3=0$$

$$(x-1)^2+2=0$$

complex valued

$$\sqrt{(x-1)^2} = \sqrt{-2}$$

could plug these in $\longrightarrow$ $x = 1 \pm \sqrt{2}\,i$

Instead, let's choose different x (for easy algebra)

$$\boxed{x=0 \qquad x=1}$$

$$7x^2-13x+13 = A(x^2-2x+3)+(Bx+C)(x-2)$$

$\boxed{@x=2}$ $\quad 7\cdot 2^2 - 13\cdot 2 + 13 = A(2^2-2\cdot 2+3) + \cancel{(B\cdot 2+C)(2-2)} \to 0$

$$15 = 28-26+13 = 3A \qquad \boxed{A=5}$$

$\boxed{@x=0}$ $\quad 7\cdot\cancel{0^2} - \cancel{13\cdot 0} + 13 = 5(\cancel{0^2}-\cancel{2\cdot 0}+3) + (\cancel{B\cdot 0}+C)(0-2)$

$$13 = 15-2C$$

$$-2 = -2C \qquad \boxed{C=1}$$

$\boxed{@x=1}$ $\quad 7 = 10 - B - 1 \longrightarrow \boxed{B=2}$

$$\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx = \int \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}\,dx$$

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx = \int \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}\,dx$$

$$= \int \frac{5}{x-2}\,dx + \int \frac{2x + 1 - 2 + 2}{x^2 - 2x + 3}\,dx$$

Try $u = x^2 - 2x + 3$

$du = (2x-2)\,dx$

$$\int \frac{5}{x-2}\,dx + \int \frac{2x-2}{x^2-2x+3}\,dx + \int \frac{3}{x^2-2x+3}\,dx$$

$$5\ln|x-2| + \int \frac{du}{u} + 3\int \frac{1}{(x-1)^2+2}\,dx$$

$v = x-1 \qquad dv = dx$

$$5\ln|x-2| + \ln|u| + 3\int \frac{dv}{v^2+2}$$

Trig subst.

$v = \sqrt{2}\tan\theta$

$dv = \sqrt{2}\sec^2\theta\,d\theta$

$$5\ln|x-2| + \ln|u| + 3\int \frac{\sqrt{2}\cancel{\sec^2\theta}\,d\theta}{(\sqrt{2}\tan\theta)^2+2} \quad = 2\cancel{\sec^2\theta}$$

$$5\ln|x-2| + \ln|u| + \frac{3}{\sqrt{2}}\int d\theta$$

$$5\ln|x-2| + \ln|u| + \frac{3}{\sqrt{2}}\theta + C$$

$u = x^2 - 2x + 3$

$v = x - 1$

$v = \sqrt{2}\tan\theta$

$\theta = \tan^{-1}\left(\frac{v}{\sqrt{2}}\right) = \tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)$

$$\boxed{5\ln|x-2| + \ln|x^2-2x+3| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C}$$

$$\boxed{5\ln|x-2| + \ln|x^2-2x+3| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C}$$

II. Repeated Quadratic Factor:

$$\int \frac{2}{x(x^2+1)^2}\,dx$$

simple linear

← repeated quadratic

$$\frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiply both sides

$$2 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

Solve for A, B, C, D, E

fast method: roots $x = 0, \pm i$,

try $x = \pm 1, \pm 2, \ldots$

Collect like terms

$$0 = \underbrace{[A+B]}_{=0}x^4 + [C+\ ]x^3 + [2A+\overset{0\,=\,D}{\cancel{B}}+\cancel{E}]x^2 + \underbrace{(C+E)}_{=0}x + \underbrace{[A-2]}_{=0}$$

$A+B=0$ $\quad 2+B=0$

$\cancel{C}^{\,0}+D=0 \rightarrow \boxed{D=0}$

$2A+D \quad =0$

$C+E=0$

$A-2=0 \rightarrow \boxed{A=2}$ $\quad \boxed{B=-2}$

$2A + D = 0$

$$2\cdot(2) + (-2) + E = 0 \longrightarrow \boxed{E = -4}$$

Numbers are wrong
Will fix

$$\int \frac{A}{x} + \frac{Bx}{x^2+1} + \int \frac{Dx+E}{(x^2+1)^2}\,dx$$

↑ u-sub?
trig-sub?

Last Example:

$$\frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiply by $x(x^2+1)^2$

$$\frac{2\cancel{x(x^2+1)^2}}{\cancel{x(x^2+1)^2}} = \frac{A}{\cancel{x}}\cancel{x}(x^2+1)^2 + \frac{Bx+C}{\cancel{x^2+1}}\cdot x(x^2+1)^{\cancel{2}} + \frac{Dx+E}{\cancel{(x^2+1)^2}}\cdot x\cancel{(x^2+1)^2}$$

$$2 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$

$$= A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2+Ex$$

$$2 = A(x^4+2x^2+1) + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$$

Now collect like terms:

$$0 = [A+B]x^4 + Cx^3 + [2A+B+D]x^2 + [C+E]x$$

$$0 = \underbrace{[A+B]}_{=0}x^4 + \underbrace{C}_{=0}x^3 + \underbrace{[2A+B+D]}_{=0}x^2 + \underbrace{[C+E]}_{=0}x + \underbrace{[A-2]}_{=0}$$

$A+B=0$ $\qquad$ $2+B=0 \rightarrow \boxed{B=-2}$

$\boxed{C=0}$

$2A+B+D=0$ $\qquad \rightarrow \qquad$ $2A+B+D=0$

$\cancel{C}+E=0 \rightarrow \boxed{E=0}$ $\qquad$ $2\cdot 2+(-2)+D=0$

$A-2=0 \rightarrow \boxed{A=2}$ $\qquad$ $\boxed{D=-2}$

$$\int \frac{2}{x(x^2+1)^2}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\,dx$$

plug in $A=2$, $B=-2$, $C=0$, $D=-2$, $E=0$

$$= \int \frac{2}{x}\,dx + \int \frac{-2x}{x^2+1}\,dx + \int \frac{-2x}{(x^2+1)^2}\,dx$$

use u-substitution

this integral is much easier now

$u = x^2+1$

$du = 2x\,dx$

$$= \int \frac{-du}{u^2}$$

$$= \left[-\frac{u^{-1}}{-1}\right] = \frac{1}{u}$$

$$= \frac{1}{x^2+1} \quad \checkmark$$