Lesson 18 - MA 166 LEC 200

Announcements:

Exam 2 Wed Mar 9 @ 6:30pm
Study Guide posted on website

8.9: Improper Integrals

(Last topic that will appear on Exam 2)

★ Warm Up: Fall 2016 Exam 2 Question #3

3. Which of the following is the partial fraction decomposition template for $\dfrac{x^3}{(x^2+4)(x-4)^2}$?

A. $\dfrac{A}{(x-4)^2} + \dfrac{B}{x^2+4}$

B. $\dfrac{A_1 x + A_2}{(x-4)^2} + \dfrac{B_1 x + B_2}{x^2+4}$

C. $\dfrac{A_1}{x-4} + \dfrac{A_2}{(x-4)^2} + \dfrac{B}{x^2+4}$

D. $\dfrac{A_1}{x-4} + \dfrac{A_2}{(x-4)^2} + \dfrac{B_1 x + B_2}{x^2+4}$

E. $\dfrac{A_1 x + A_2}{(x-4)^2} + \dfrac{B}{x^2+4}$

simple quadratic

repeated linear

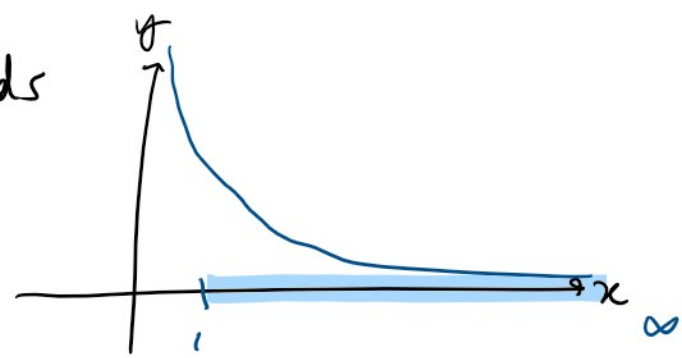
$$\frac{Cx+D}{x^2+4} + \frac{A}{x-4} + \frac{B}{(x-4)^2}$$

I. Improper Integrals:

1) Infinite Intervals
2) Unbounded Integrands

1) Infinite Intervals

$$\int_1^\infty \frac{1}{x^2}\,dx$$



Q: Does this integral have a finite value?
→ converges (if yes)

Rewrite as a limit of definite integrals

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim \left[\frac{x^{-1}}{-1} \right]^b = \lim \left[-\frac{1}{b} - \left(-\frac{1}{1}\right) \right] = 1$$

$$= \lim_{b\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^b = \lim_{b\to\infty}\left[\underbrace{\frac{-1}{b}}_{\to 0 \text{ as } b\to\infty} - \left(\frac{-1}{1}\right)\right] = 1$$

$$\boxed{\int_1^\infty \frac{1}{x^2}\,dx = 1}$$

The integral converges

Alternatively $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \left[\ln(x)\right]_1^b = \lim_{b\to\infty}\left[\ln(b) - \cancelto{0}{\ln(1)}\right] = +\infty$$

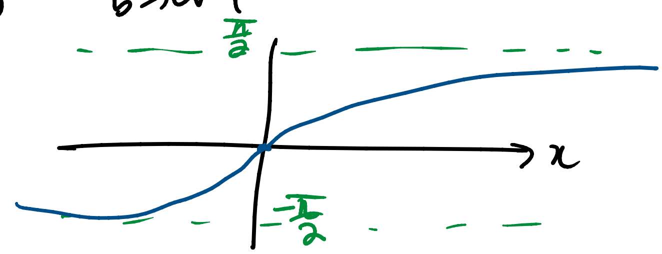
diverges

"The integral diverges"

Ex: $$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty}\left[\tan^{-1}(x)\right]_0^b = \lim_{b\to\infty}\left[\tan^{-1}(b) - \cancelto{0}{\tan^{-1}(0)}\right]$$

$$= \frac{\pi}{2}$$



the integral converges

Q: For what values of p does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$$

$\boxed{\text{Case 1}}$ $p = 1$ $\displaystyle\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(b) = +\infty$

$\boxed{\text{case 1}}$ $p=1$ $\quad \lim\limits_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim\limits_{b\to\infty} \ln(b) = +\infty$

Diverges

Otherwise: $\lim\limits_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \lim\limits_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$

$= \lim\limits_{b\to\infty} \left(\frac{b^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right]$

$= \lim\limits_{b\to\infty} \frac{1}{1-p}\left[b^{-p+1} - 1 \right]$

$-p+1 > 0$

$\xrightarrow[\text{as } b\to\infty]{} +\infty$

$-p+1 < 0$

$\frac{1}{b^{p-1}} \xrightarrow[\text{as } b\to\infty]{} 0$

$p<1$	$p=1$	$p>1$
diverges	diverges	converges

If $p \le 1$ $\quad \int_1^\infty \frac{1}{x^p}\,dx$ diverges

$p > 1$ $\quad \int_1^\infty \frac{1}{x^p}\,dx$ converges

check: $p = \frac{1}{2}$

$\int_1^\infty \frac{1}{x^{1/2}}\,dx = \lim\limits_{b\to\infty} \int_1^b x^{-1/2}\,dx$

$$\int_1^{\infty} \frac{1}{x^{1/2}}\,dx = \lim_{b\to\infty} \int_1^b$$

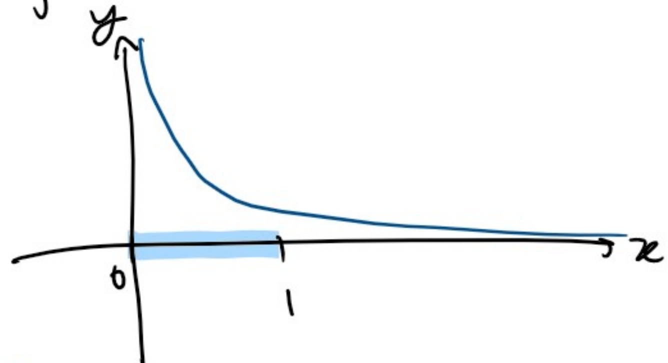
$$= \lim_{b\to\infty} \left[\frac{x^{1/2}}{1/2}\right]_1^b = \lim_{b\to\infty} \left[\underbrace{2x^{1/2}}_{\to +\infty \text{ as } b\to\infty} - 2\cdot 1^{1/2}\right]$$

diverges ✓

2) Unbounded integrand

Q: What if the integrand is undefined at one point?

Ex: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$



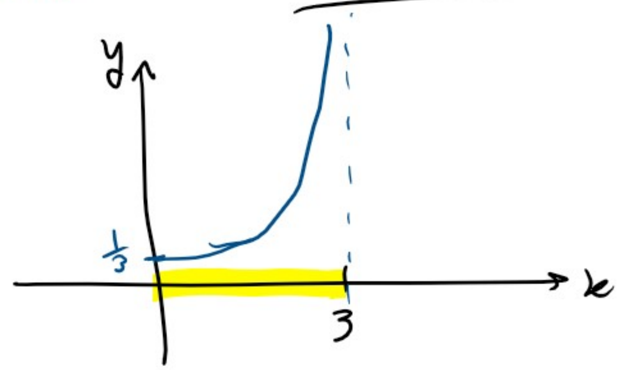
$\frac{1}{\sqrt{x}}$ is undefined at $x=0$

$$= \lim_{c\to 0^+} \int_c^1 \frac{dx}{\sqrt{x}} = \lim_{c\to 0^+} \left[\frac{x^{1/2}}{1/2}\right]_c^1$$

$$= \lim_{c\to 0^+} \left[2\cdot 1^{1/2} - \underbrace{2\cdot c^{1/2}}_{\to 0 \text{ as } c\to 0^+}\right] = \boxed{2}$$

integral is defined

Ex: $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$



$$= \lim_{c\to 3^-} \int_0^c \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \lim_{c \to 3^-} \left[\sin^{-1}\left(\frac{x}{3}\right) \right]_0^c$$

$\sin^{-1}(0) = x$

$0 = \sin(x)$

$x = 0$

$$= \lim_{c \to 3^-} \left[\sin^{-1}\left(\frac{c}{3}\right) - \underbrace{\sin^{-1}\left(\frac{0}{3}\right)}_{0} \right]$$

$\sin^{-1}\left(\frac{c}{3}\right) = x$

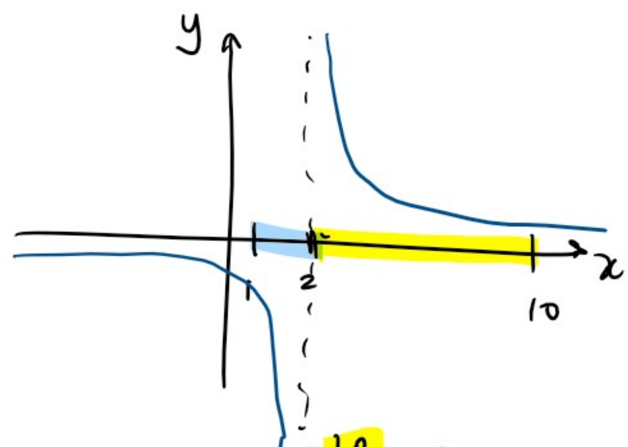
in limit

$\sin^{-1}(1) = x$?

$1 = \sin(x) \longrightarrow x = \frac{\pi}{2}$

$$= \left[\frac{\pi}{2} - 0 \right] = \boxed{\frac{\pi}{2}}$$ integral converges

<u>Ex</u>: Evaluate $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$



integrand undefined @ $x = 2$.

$$= \int_1^{10} \frac{dx}{(x-2)^{1/3}} = \lim_{c \to 2^-} \int_1^c \frac{dx}{(x-2)^{1/3}} + \lim_{c \to 2^+} \int_c^{10} \frac{dx}{(x-2)^{1/3}}$$

$$= \lim_{c \to 2^-} \left[\frac{(x-2)^{2/3}}{2/3} \right]_1^c + \lim_{c \to 2^+} \left[\frac{(x-2)^{2/3}}{2/3} \right]_c^{10}$$

$$= \frac{3}{2} \lim_{c \to 2^-} \left[\underbrace{(c-2)^{2/3}}_{\to 0^-} - \overbrace{(\underbrace{1-2}_{-1})^{2/3}}^{\to +1} \right]$$

as $c \to 2^-$

$$+ \frac{3}{2} \lim_{c \to 2^+} \left[\underbrace{(10-2)^{2/3}}_{8^{2/3}} - (c-2)^{2/3} \right]$$

$$= \frac{3}{2}\left[0-(+1)\right] + \frac{3}{2}\left[\overset{''4}{4} - 0\right]$$

$$= \frac{-3}{2} + \frac{3}{2}\cdot 4 = \frac{12-3}{2} = \boxed{\frac{9}{2}}$$ integral converges

Q: Does $\int_1^\infty e^{-x^2}\,dx$ converge?

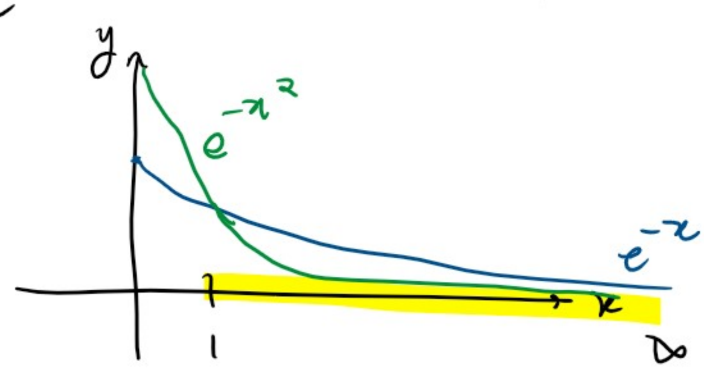
Trouble $\int e^{-x^2}\,dx = ?$

Solution: Let's compare this integral to something we know

$$x \le x^2 \quad \text{on } [1,\infty]$$

$$0 \le e^{-x^2} \le e^{-x} \quad \text{on } [1,\infty]$$

so:



$$0 \le \int_1^b e^{-x^2}\,dx \le \int_1^b e^{-x}\,dx$$

$$\lim_{b\to\infty}\int_1^b e^{-x^2}\,dx \le \lim_{b\to\infty}\int_1^b e^{-x}\,dx$$

$$\left[-x \right]^b$$

$$\lim_{b\to\infty}\int_1 e \quad dx \qquad\qquad \lim_{b\to\infty}\int_1$$

$$= \lim_{b\to\infty}\left[\frac{e^{-x}}{-1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[-e^{-b} + (+e^{-1})\right]$$

$$= \lim_{b\to\infty}\left[\underbrace{-\frac{1}{e^b}}_{\to 0} + \frac{1}{e}\right] = \frac{1}{e}$$

$$0 \le \int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx = \frac{1}{e}$$

so yes the integral converges

<u>Comparison Test</u>