Lesson 18 - MA 166 LEC 286

8.9: Improper Integrals

(Last topic that will appear on Exam 2)

Announcements:

Exam 2 Wed Mar 9 @ 6:30pm

Study Guide posted on website

Warm Up: Fall 2016 Exam 2 Question #3

3. Which of the following is the partial fraction decomposition template for $\dfrac{x^3}{(x^2+4)(x-4)^2}$?

A. $\dfrac{A}{(x-4)^2} + \dfrac{B}{x^2+4}$

B. $\dfrac{A_1x + A_2}{(x-4)^2} + \dfrac{B_1x + B_2}{x^2+4}$

C. $\dfrac{A_1}{x-4} + \dfrac{A_2}{(x-4)^2} + \dfrac{B}{x^2+4}$

D. $\boxed{\dfrac{A_1}{x-4} + \dfrac{A_2}{(x-4)^2} + \dfrac{B_1x + B_2}{x^2+4}}$

E. $\cancel{\dfrac{A_1x + A_2}{(x-4)^2}} + \dfrac{B}{x^2+4}$

simple quadratic

repeated linear

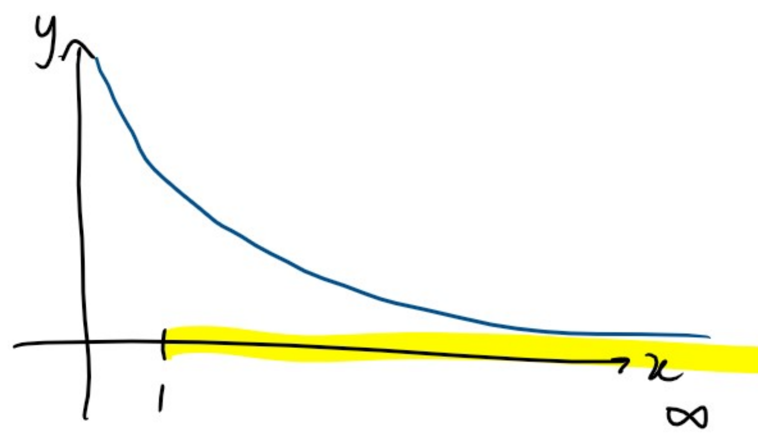
$$\frac{Cx+D}{x^2+4} + \frac{A}{x-4} + \frac{B}{(x-4)^2}$$

I. Improper Integrals:

2 types

1) Infinite Intervals

$$\int_1^\infty \frac{1}{x^2}\,dx = ?$$



Q: Does this integral have a finite value?

If yes → converges

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

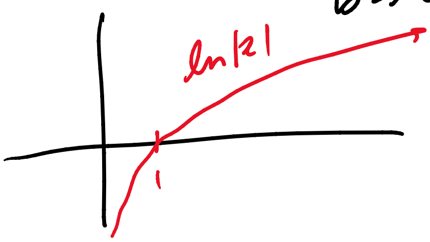
$$= \lim_{b\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{b} - \left(-\frac{1}{1}\right)\right] = 1$$

$$= \lim_{b\to\infty}\left[\frac{x^{-1}}{-1}\right]_1^b = \lim_{b\to\infty}\left(-\underbrace{\frac{1}{b}}_{\to 0 \text{ as } b\to\infty} + 1\right)$$

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$ "The integral converges"

Alternatively: $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty}\Big[\ln|x|\Big]_1^b = \lim_{b\to\infty}\left[\ln|b| - \cancelto{0}{\ln|1|}\right]$$



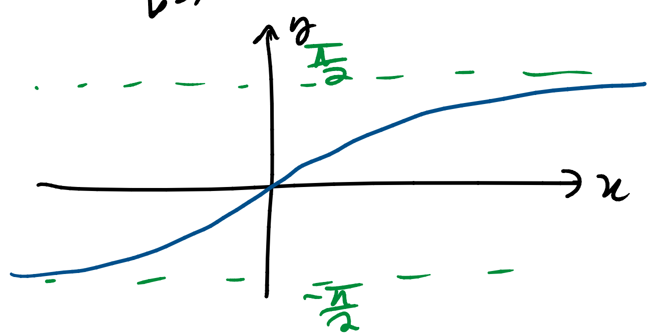
$$= +\infty$$

so "the integral diverges"

<u>Ex</u>: $$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty}\Big[\tan^{-1}(x)\Big]_0^b = \lim_{b\to\infty}\left[\tan^{-1}(b) - \cancelto{0}{\tan^{-1}(0)}\right]$$

$$= \boxed{\frac{\pi}{2}}$$



"the integral converges"

<u>Q</u>: For what values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$$

if $p=1$

$\boxed{\text{case 1}}$ $p=1$ $$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|b| - \ln|1|\right]$$ diverges

$\boxed{\text{case 1}}$ $p=1$ $\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}(\ln|b| - \ln|1|)$

Diverges

only true if $p \neq 1$

otherwise:

$$\lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{b^{-p+1}}{-p+1} - \frac{1^{-b+1}}{-p+1}\right]$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left[b^{-p+1} - 1\right]$$

$\boxed{\text{case 2}}$

$-p+1 > 0$

$\alpha = -p+1 > 0$

$\lim_{b\to\infty}\left[b^{\alpha} - 1\right] = +\infty$

Diverges

$\boxed{\text{case 3}}$

$-p+1 < 0$

$-\alpha = -p+1 < 0$

$\lim_{b\to\infty}\left[\frac{1}{b^{\alpha}} - 1\right] = -1$

Converges

$\boxed{p<1}$		$\boxed{p>1}$
$+p\ -p+1 > 0\ +p$	$p=1$	$+p\ -p+1 < 0\ +p$
Diverges	Diverges	Converges

check: $\int_1^{\infty}\frac{1}{x^{1/2}}\,dx$ — diverge

$\left[x^{1/2}\right]^b$

$\int_1 \; x^{1/2}$

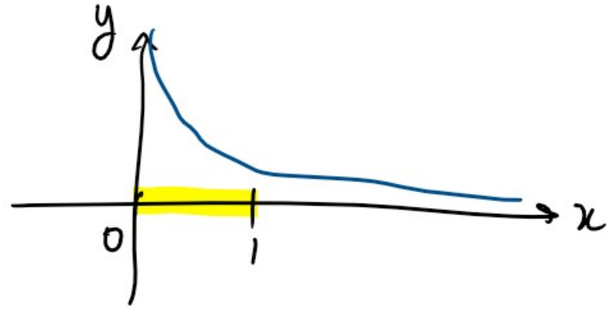
$$= \lim_{b\to\infty} \int_1^b x^{-1/2}\,dx = \lim_{b\to\infty}\left[\frac{x^{1/2}}{1/2}\right]_1^b$$

$$= \lim_{b\to\infty}\left[2b^{1/2} - 2\cdot 1^{1/2}\right] = +\infty$$

The integral diverges ✓

2) Unbounded Integrands

Q: What if the integrand is undefined at one point?



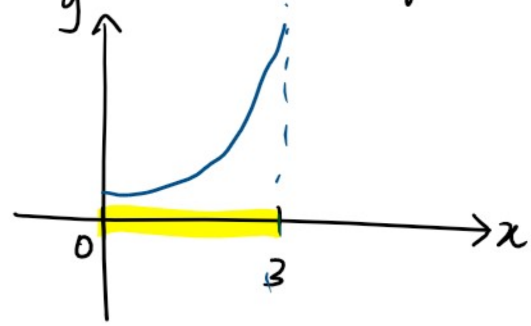
Ex: $\int_0^1 \frac{dx}{\sqrt{x}}$

$\frac{1}{\sqrt{x}}$ is undefined @ $x=0$

$$= \lim_{c\to 0^+} \int_c^1 \frac{dx}{\sqrt{x}} = \lim_{c\to 0^+}\left[\frac{x^{1/2}}{1/2}\right]_c^1$$

$$= \lim_{c\to 0^+}\left[2\sqrt{1} - \underbrace{2\sqrt{c}}_{\to 0 \text{ as } c\to 0^+}\right] = \boxed{2}$$

The integral converges.



Ex: $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

1 is undefined @ $x=3$

integrand is undefined @ $x=3$

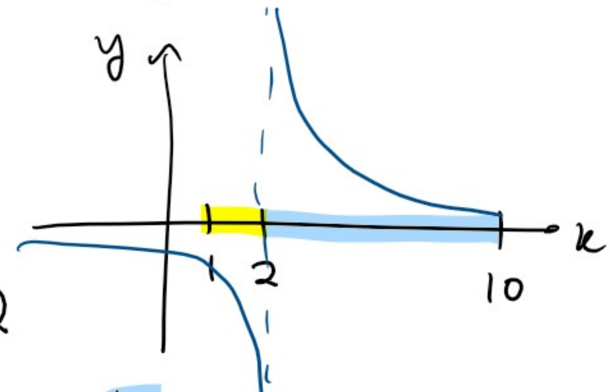
$$= \lim_{c\to 3^-} \int_0^c \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{c\to 3^-} \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^c$$

$$= \lim_{c\to 3^-} \left[\sin^{-1}\left(\frac{c}{3}\right) - \cancel{\sin^{-1}\left(\frac{0}{3}\right)}^{\,0}\right]$$

$\sin\left(\frac{\pi}{2}\right) = 1$ $\qquad$ $\sin(0) = 0$

$$= \boxed{\frac{\pi}{2}}$$ Integral converges

Evaluate $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$



Integrand is undefined @ $x=2$

$$= \lim_{c\to 2^-} \int_1^c \frac{dx}{(x-2)^{1/3}} + \lim_{d\to 2^+} \int_d^{10} \frac{dx}{(x-2)^{1/3}}$$

$$= \lim_{c\to 2^-} \left[\frac{(x-2)^{2/3}}{2/3}\right]_1^c + \lim_{d\to 2^+} \left[\frac{(x-2)^{2/3}}{2/3}\right]_d^{10}$$

$$= \frac{3}{2}\lim_{c\to 2^-} \left[(c-2)^{2/3} - (1-2)^{2/3}\right] + \frac{3}{2}\lim_{d\to 2^+} \left[(10-2)^{2/3} - (d-2)^{2/3}\right]$$

$$= \frac{3}{2}\left[0^{2/3} - (-1)^{2/3}\right] + \frac{3}{2}\left[8^{2/3} - 0^{2/3}\right]$$

$$= \frac{3}{2}\left[0^{2/3} - (-1)^{2/3}\right] + \frac{3}{2}\left[8^{2/3} - 0^{2/3}\right]$$

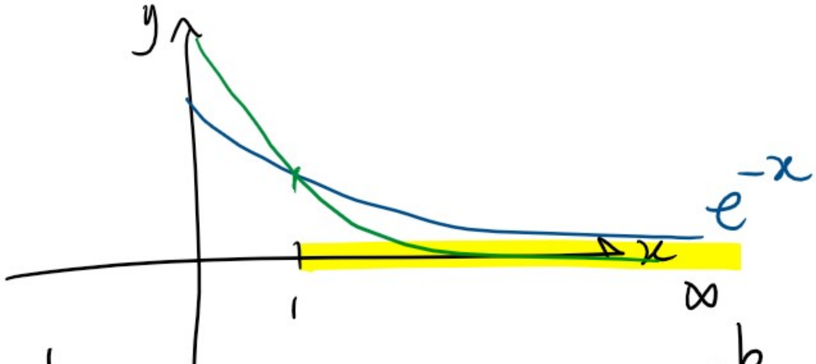
$$= \frac{3}{2}(-1) + \frac{3}{2}\cdot 4 = \frac{12-3}{2} = \boxed{\frac{9}{2}}$$ the integral converges.

Q: Does $\int_1^\infty e^{-x^2}\,dx$ converge?

$$= \lim_{b\to\infty} \int_1^b e^{-x^2}\,dx =$$ Trouble $\int e^{-x^2}\,dx = ?$

Solution: Let's compare w/ something we know how to evaluate

$$0 \le e^{-x^2} \le e^{-x}$$ on x $[1, \infty]$



$$0 \le \lim_{b\to\infty} \int_1^b e^{-x^2}\,dx \le \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{e^{-x}}{-1}\right]_1^b$$

$$= \lim \left[-e^{-b} - (-e^{-1})\right]$$

$$= \lim_{b\to\infty}\left[-e^{-b} - (-e^{-1})\right]$$

$$= \lim_{b\to\infty}\left[\frac{-1}{e^{b}} + \frac{1}{e}\right] = \frac{1}{e}$$

$\frac{-1}{e^b} \to 0$

$$0 \le \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx \le \frac{1}{e}$$

the integral converges

<u>Comparison Test</u>