Lesson 19 - MA 166 LEC 200

Sequence and its Limit (10.1 & 10.2)

Announcements:

Exam 2 is Wed Mar 9 @ 6:30pm
Students are assigned seats

GOALS:

- Find terms of sequences
- Determine the limit of a sequence

WARM UP: Fall 2019 Exam 2 Question #6

6. $\int_0^\infty \sin u\, du =$

A. 0
B. 1
C. -1
D. 2
E. The integral diverges.

$$\lim_{b\to\infty} \int_0^b \sin(u)\,du$$

$$= \lim_{b\to\infty} \Big[-\cos(u)\Big]_0^b$$

$$= \lim_{b\to\infty} \Big[-\cos(b) + \cancel{\cos(0)}^{1}\Big] = \text{DNE}$$

diverges

☆ Students are assigned seats for Exam 2.

I. Sequences:

Def: A sequence $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

Sometimes we write $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}$

here $n = 1, 2, 3, 4, \ldots$ (natural numbers/positive integers)

Examples:

i) $a_n = \frac{1}{2^n}$

Examples.

(1) $a_n = \frac{1}{2^n}$

$a_1 = \frac{1}{2^1} = \frac{1}{2}$, $a_2 = \frac{1}{2^2}$, $a_3 = \frac{1}{2^3}$, ...

(2) $a_n = \frac{(-1)^n}{n^2+1}$

$a_1 = \frac{(-1)^1}{1^2+1} = \frac{-1}{2}$, $a_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5}$

$a_3 = \frac{(-1)^3}{3^2+1} = \frac{-1}{10}$, ...

A sequence can be defined by a recurrence relation.

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 2 \cdot a_n + 1 \end{cases}$$

$a_1 = 1$

$a_2 = 2 \cdot a_1 + 1$
$= 2 \cdot 1 + 1 = 3$

$a_3 = 2 \cdot a_2 + 1$
$= 2 \cdot 3 + 1 = 7$

$\vdots$

GOAL: Find the limit of a sequence

$\lim\limits_{n \to \infty} a_n$

Examples:

(1) $a_n = \frac{(-1)^n}{n^2+1}$

n

(1) $a_n = \dfrac{(-1)}{n^2+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2+1}$$

each term is bounded

$$\frac{-1}{n^2+1} \le a_n \le \frac{+1}{n^2+1}$$

$$\frac{-1}{n^2+1} \xrightarrow{n\to\infty} 0 \qquad a_n \xrightarrow{n\to\infty} 0 \qquad \frac{+1}{n^2+1} \xrightarrow{n\to\infty} 0$$

Squeeze Thm

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$$

"The sequence converges to zero"

Ex (2): $a_n = \cos(n\pi)$

$a_1 = \cos(\pi) = -1$

$a_2 = \cos(2\pi) = +1$

$a_3 = \cos(3\pi) = -1$

$a_4 = \cos(4\pi) = +1$

$\vdots$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n\pi) = \text{DNE}$$

"The sequence diverges"

Ex (3):

$$\left.\begin{aligned} a_1 &= 1 \\ a_{n+1} &= -2a_n \end{aligned}\right\} \text{ recurrence relation}$$

Can we write this as explicit formula?

Q: Can we write this as explicit formula?

$$a_1 = 1$$
$$a_2 = -2 \cdot 1 = -2$$
$$a_3 = -2 \cdot (-2) = (-2)^2$$
$$a_4 = -2(-2)^2 = (-2)^3$$

$$\boxed{a_n = (-2)^{n-1}}$$

explicit formula

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-2)^{n-1} = \text{DNE}$$

the sequence diverges

Ex: $a_n = \dfrac{4n^3}{n^3+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^3/n^3}{(n^3+1)/n^3} = \lim_{n\to\infty} \frac{4}{1+\left(\frac{1}{n^3}\right)} = 4$$

The sequence converges to 4.

Ex: $a_n = \left(\dfrac{n+5}{n}\right)^n$

Q: Does this sequence converge? If so, what is its limit?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+5}{n}\right)^n = e^5$$

$$= \lim_{n\to\infty} \left(1 + \frac{5}{n}\right)^n \sim 1^\infty$$

$$y = \left(1 + \frac{5}{n}\right)^n$$

$$\ln y = \ln\left[\left(1+\frac{5}{n}\right)^n\right] = n \ln\left(1+\frac{5}{n}\right)$$

$$\ln y = \ln\left(\left(1+\frac{5}{n}\right)^n\right)$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \underset{\substack{\downarrow \\ \infty}}{n} \underbrace{\ln\left(1+\frac{5}{n}\right)}_{\to 0}$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{5}{n}\right)}{\left(\frac{1}{n}\right)} \begin{array}{l} \to 0 \\ \to 0 \end{array}$$

Apply L'Hopital's Rule

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{5}{n}}\right)\cdot\left(\frac{+5}{\cancel{n^2}}\right)}{\frac{+1}{\cancel{n^2}}}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{5}{1+\frac{5}{n}} = 5$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln y \to 5} = \boxed{e^5}$$

Geometric Sequences:

Ex: ① $a_n = \left(\frac{1}{2}\right)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$$

② $a_n = \left(\frac{-1}{2}\right)^n$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{-1}{2}\right)^n = 0$

③ $a_n = 3^n$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} 3^n = +\infty$

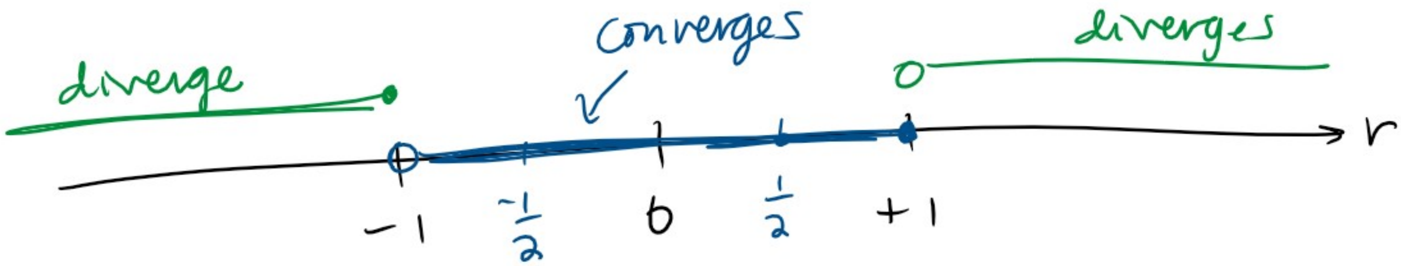
④ $a_n = (-3)^n$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} (-3)^n = \text{DNE}$

⑤ $a_n = (1)^n$ $\quad \lim_{n\to\infty} (1)^n = 1$

(5) $a_n = (1)^n \qquad \lim_{n\to\infty} (1)^n = 1$

(6) $a_n = (-1)^n \qquad \lim_{n\to\infty} (-1)^n = \text{DNE}$

Geometric: $a_n = r^n$ where r is a real number



$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=+1 \\ \text{diverges} & \text{if } |r|>1 \\ & \quad r=-1 \end{cases}$$

Thm: (Growth Rates of Sequences)

according increasing growth rates

slowest growth

$$\{\ln^q(n)\} \ll \{n^p\} \ll \{n^p \ln^r(n)\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}$$

fastest growth

where $p, q, r, s, b > 1$

Ex: $\lim_{n\to\infty} \dfrac{n^2}{n!} = 0$ $\quad n^p,\ p=2$

$n^2 \ll n!$

$$\lim_{n\to\infty} \frac{n!}{n^2} = \infty \quad \text{diverges}$$