Announcements:

Exam 2 is Wed Mar 9 @ 6:30pm
Students are assigned seats

Sequence and its Limit (10.1 & 10.2)

GOALS:

- Find terms of sequences
- Determine the limit of a sequence

WARM UP: Fall 2019 Exam 2 Question #6

6. $\int_0^\infty \sin u \, du =$

A. 0

B. 1

C. -1

D. 2

E. The integral diverges.

$$\lim_{b\to\infty} \int_0^b \sin(u)\,du$$

$$= \lim_{b\to\infty} \Big[-\cos(u)\Big]_0^b$$

$$= \lim_{b\to\infty} \Big[-\cos(b) + \cancel{\cos(0)}^{1}\Big] = \text{DNE}$$

★ Students are assigned seats for exam 2

I Sequences:

Def: A sequence $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

Sometimes we write:

$$\{a_n\}_{n=1}^{\infty}, \quad \{a_n\}$$

here $n = 1, 2, 3, 4, \ldots$ (natural numbers / positive integers)

Examples:

$a_n = \frac{1}{\,}$

Examples:

(1) $a_n = \frac{1}{2^n}$

$a_1 = \frac{1}{2^1} = \frac{1}{2}, \quad a_2 = \frac{1}{2^2}, \quad a_3 = \frac{1}{2^3}, \ldots$

(2) $a_n = \frac{(-1)^n}{n^2+1}$

$a_1 = \frac{(-1)^1}{1^2+1} = \frac{-1}{2}, \quad a_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5}$

$a_3 = \frac{(-1)^3}{3^2+1} = -\frac{1}{10}, \ldots$

A sequence can be defined by a <u>recurrence</u> <u>relation</u>

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 2a_n + 1 \end{cases}$$

$a_1 = 1$

$a_2 = 2 \cdot a_1 + 1 = 2 \cdot 1 + 1 = 3$

$a_3 = 2 \cdot a_2 + 1 = 2 \cdot 3 + 1 = 7$

$\vdots$

<u>GOAL</u>: Find the limit of a sequence

$\lim_{n\to\infty} a_n$

<u>Examples</u>:

(1) $a_n = \frac{(-1)^n}{n^2+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2+1}$

... sequence is bounded

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ldots}{n^2+1}$$

each term in the sequence is bounded

$$-\frac{1}{n^2+1} \le a_n \le \frac{+1}{n^2+1} \qquad \begin{pmatrix} n=2 \\ n=3 \end{pmatrix}$$

$$\begin{array}{ccccc} -\frac{1}{n^2+1} & \le & a_n & \le & \frac{+1}{n^2+1} \\ \big\downarrow{\scriptstyle n\to\infty} & & \big\downarrow{\scriptstyle n\to\infty} & & \big\downarrow{\scriptstyle n\to\infty} \\ 0 & & 0 & & 0 \end{array}$$

$\leftarrow$ Squeeze Theorem

$$\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$$

"The sequence converges to zero"

Example 2: $a_n = \cos(n\pi)$

$a_1 = \cos(\pi) = -1$

$a_2 = \cos(2\pi) = +1$

$a_3 = \cos(3\pi) = -1$

$a_4 = \cos(4\pi) = +1$

$\vdots$

$$\lim_{n\to\infty} a_n = \text{DNE}$$

"The sequence diverges"

Example 3: $\begin{cases} a_1 = 1 \\ a_{n+1} = -2a_n \end{cases}$ recurrence relation

Q: Can we write this as an explicit formula?

$a_1 = 1 = (-2)^0$

$a_2 = -2 \cdot 1 = -2$

$a_3 = -2 \cdot (-2) = (-2)^2$

$a_4 = -2 \cdot (-2)^2 = (-2)^3$

$a_4 = \ldots$

$$a_n = (-2)^{n-1}$$

$$\lim_{n\to\infty} (-2)^{n-1} = DNE$$ sequence diverges

Example 4: $a_n = \frac{4n^3}{n^3+1}$

$$\lim_{n\to\infty} \frac{4n^3/n^3}{(n^3+1)/n^3} = \lim_{n\to\infty} \frac{4}{1+\left(\frac{1}{n^3}\right)} = 4$$

"The limit of the sequence is 4"

Example 5: $a_n = \left(\frac{n+5}{n}\right)^n$

Q: Does this sequence converge? And if so, what is the limit?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+5}{n}\right)^n = \lim_{n\to\infty} \Big(\underbrace{1+\frac{5}{n}}_{\to 1}\Big)^{n \to \infty} = e^5$$

indeterminate

Let $y = \left(1+\frac{5}{n}\right)^n$

$$\ln y = \ln\left[\left(1+\frac{5}{n}\right)^n\right] = n\ln\left(1+\frac{5}{n}\right)$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \overset{\to\infty}{n} \ln\Big(1+\overset{\to 0}{\frac{5}{n}}\Big)$$

$$= \lim \frac{\ln\left(1+\frac{5}{n}\right) \to 0}{}$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{5}{n}\right) \to 0}{\left(\frac{1}{n}\right) \to 0} \quad \frac{0}{0}$$

L'Hopital's Rule

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{5}{n}} \cdot \left(+\frac{5}{\cancel{n^2}}\right)}{\left(\frac{+1}{\cancel{n^2}}\right)}$$

$$\lim_{n\to\infty} \overbrace{\ln\left(\left(\frac{n+5}{n}\right)^n\right)}^{\ln y} = \lim_{n\to\infty} \frac{5}{1+\frac{5}{n}} = 5$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln y \xrightarrow{n\to\infty} 5} = \boxed{e^5}$$

II. Geometric Sequences:

$a_n = r^n$ where r is a real number

① $a_n = \left(\frac{1}{2}\right)^n$ $\qquad \lim_{n\to\infty} \frac{1}{2^n} = 0$

② $a_n = \left(-\frac{1}{2}\right)^n$ $\qquad \lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$

③ $a_n = 3^n$ $\qquad \lim_{n\to\infty} 3^n = +\infty$

Q: For what values of r does the geometric sequence have a limit?

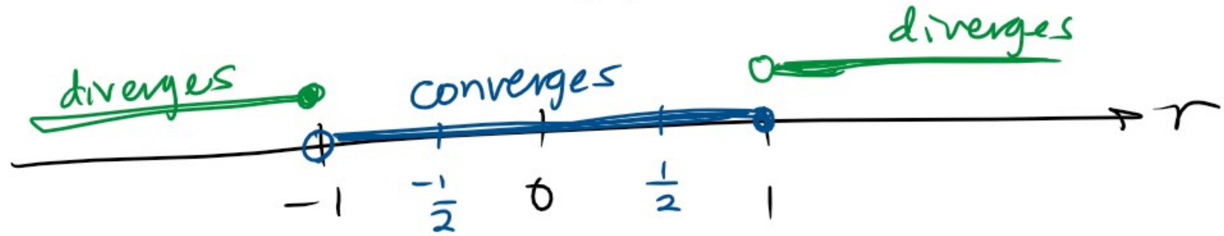
④ $a_n = (-3)^n$ $\qquad \lim_{n\to\infty} (-3)^n = \text{DNE}$

$a_1 = 1,\ a_2 = 1,\ a_3 = 1, \ldots$

⑤ $a_n = 1^n$ $\qquad \lim_{n\to\infty} a_n = 1$

(5) $a_n = 1^n$ $\qquad\qquad \lim\limits_{n\to\infty} a_n = 1$

(6) $a_n = (-1)^n$ $\qquad\qquad \lim\limits_{n\to\infty} (-1)^n = \text{DNE}$



$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=+1 \\ \text{diverges} & \text{if } r=-1,\ |r|>1 \end{cases}$$

<u>Theorem</u>: (Growth Rates of Sequences)

order according to increasing growth rates

Let $p, q, r, s, b > 1$

$$\{\ln^q(n)\} \ll \{n^p\} \ll \{n^p \ln^r(n)\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}$$

slowest growth

fastest growth

n^p where $p=2$

<u>Example</u>: $\displaystyle\lim_{n\to\infty} \frac{n^2}{n!} = 0$

$$\lim_{n\to\infty} \frac{n!}{n^2} = \infty$$