Introduction to Series (10.1 & 10.3)

Announcements:

Exam 2 on Wed Mar 9 @ 6:30pm

students are assigned seats in ELLT

Warm Up: Spring 2014 Exam 2 Question #12 (Restated for Clarity)

Which of the following sequences converge?

(1) $\left\{ a_n = \frac{2n}{3n+1} \right\}$

(2) $\left\{ a_n = \cos(n\pi) \right\}$

(3) $\left\{ a_n = n \sin\left(\frac{1}{n}\right) \right\}$

A. (1), (2), & (3)

B. (2) & (3)

$\boxed{\text{C. (1) \& (3)}}$

D. (1) & (2)

E. (1)

(1) $\lim_{n\to\infty} \frac{2n}{3n+1} = \lim_{n\to\infty} \frac{2}{3+\frac{1}{n}} = \frac{2}{3}$

(2) diverges

(3) $\lim_{n\to\infty} \underbrace{n}_{\to\infty} \underbrace{\sin\left(\frac{1}{n}\right)}_{\to 0} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right) \to 0}{\frac{1}{n} \to 0}$ L'Hopital's

$= \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cancel{\left(\frac{-1}{n^2}\right)}}{\cancel{\left(\frac{-1}{n^2}\right)}} = 1$

I. Series:

Def: Given a sequence $\{a_1, a_2, a_3 \ldots\}$ the sum of its terms

$$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$$

is called an infinite series

The sequence of partial sums $\{S_n\}$

The sequence of partial sums $\{S_n\}$ associated is

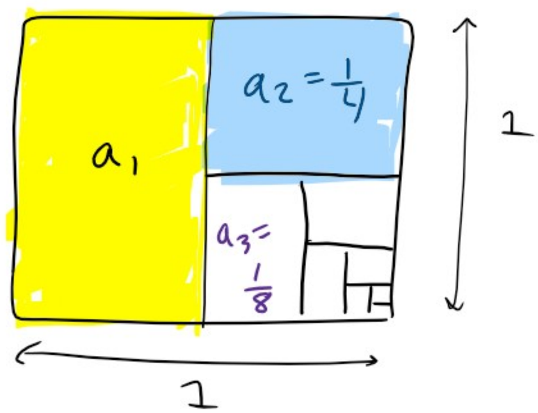
$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$

$$S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k$$

If $\{S_n\}$ has a limit L, the infinite series converges to L.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L$$

If $\{S_n\}$ diverges, then the infinite series also diverges



Examples:

① $a_n = \frac{1}{2^n}$

$a_1 = \frac{1}{2}$ $\quad S_1 = \frac{1}{2}$

$a_2 = \frac{1}{2^2}$ $\quad S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$a_3 = \frac{1}{8}$ $\quad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$a_4 = \frac{1}{16}$ $\quad S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

$$S_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

$$\lim S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$$

so $$\boxed{\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} S_n = 1}$$

Example 2:

Consider the series:

$$0.9 + 0.09 + 0.009 + 0.0009 + \cdots$$

Here $a_n = \frac{9}{10^n} = 9(0.1)^n$

$$S_n = \sum_{k=1}^{n} 9(0.1)^k$$

$$S_1 = 0.9$$

$$S_2 = 0.9 + 0.09 = 0.99$$

$$S_3 = 0.9 + 0.09 + 0.009 = 0.999$$

$$\lim_{n\to\infty} S_n = 1$$

so $$\sum_{k=1}^{\infty} 9(0.1)^k = 1$$

II. Geometric Series:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$= \sum_{k=0}^{n-1} ar^k$$

geometric sum

r – ratio

$$\sum^{\infty} ar^k \leftarrow$$ geometric series

$$\sum_{k=0}^{\infty} ar^k \leftarrow \underline{\underline{\text{geometric series}}}$$

starts at zero $\rightarrow$

<u>Ex</u>: $$\sum_{k=0}^{\infty} 0.9(0.1)^k = 0.9 + 0.09 + 0.009 + \dots$$

$$a = 0.9 \qquad r = 0.1$$

$$0.9(\cancel{0.1})^{0} + \underbrace{0.9(0.1)^1}_{0.09} + \dots$$
$$= 0.9 + 0.09 + \dots$$

<u>Want</u>: $$\lim_{n\to\infty} S_n \overset{?}{=} L \overset{?}{=} \sum_{k=0}^{\infty} ar^k$$

$$S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$$

compare

$$- \quad rS_n = r(a + ar + \dots + ar^{n-2} + ar^{n-1})$$
$$= ar + ar^2 + \dots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}} - \{\cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}} + ar^n\}$$

$$S_n - rS_n = a - ar^n$$

$$(1-r)S_n = a(1-r^n) \leftarrow \text{true if } |r| \neq 1$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right)$$

$r^n \rightarrow 0$ if $|r| < 1$

$$= \begin{cases} \dfrac{a}{1-r} & |r| < 1 \\ \text{diverges} & r=1,\ r=-1 \\ \text{DNE} & |r| > 1 \end{cases}$$

If $r=1$, $\displaystyle\sum_{k=0}^{\infty} a(1)^k = \underbrace{a + a + a + a + \cdots}_{\text{diverge}}$

If $r=-1$ $\displaystyle\sum_{k=0}^{\infty} a(-1)^k = \underbrace{a - a + a - a + a - a + \cdots}_{\text{diverges}}$

$$\boxed{\sum_{k=0}^{\infty} ar^k = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}}$$

because $|r| = \left|\frac{1}{e}\right| < 1$

Example:

(1) $\displaystyle\sum_{k=0}^{\infty} e^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{e}\right)^k = \frac{a}{1-r} = \frac{(1)e}{\left(1-\frac{1}{e}\right)e}$

$a=1$ $r = \frac{1}{e} < 1$ $= \boxed{\dfrac{e}{e-1}}$

(2) $\displaystyle\sum_{k=2}^{\infty} 3(-0.75)^k$

should start @ $k=0$

$= 3(-0.75)^2 + 3(-0.75)^3 + 3(-0.75)^4 + \cdots$

$= a + ar + ar^2 + \cdots$

$\boxed{a = 3(-0.75)^2}$

$$\boxed{a = 3(-0.75)^2}$$

$$ar = 3(-0.75)^3 = \underbrace{3(-0.75)^2}_{a}(-0.75)$$

$$\sum_{k=0}^{\infty} \overbrace{\left[3(-0.75)^2\right]}^{a} (-0.75)^k = \sum_{k=0}^{\infty} ar^k \quad \rightarrow r = -0.75$$

$$|r| = |-0.75| < 1$$

$$= \frac{a}{1-r}$$

$$\frac{3\left(\frac{9}{\cancel{16}\,4}\right)}{\frac{7}{\cancel{4}}} = \frac{3\left(\frac{-3}{4}\right)^2}{1+\frac{3}{4}} = \frac{3(-0.75)^2}{1-(-0.75)}$$

$$= \frac{3 \cdot 9}{7 \cdot 4} = \boxed{\frac{27}{28}}$$

III. Telescoping Series:

Example: $a_k = \dfrac{1}{k(k+1)}$ $\qquad \displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = ?$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$$

Instead, use partial fractions

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}$$

$k=1$

Telescoping series

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots$$

$$+ \cdots \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$