Lesson 20 - MA 166 LEC 286

Introduction to Series (10.1 & 10.3)

Announcements:
Exam 2 on Wed Mar 9 @ 6:30pm
Students are assigned seats in ELLT

Warm Up: Spring 2014 Exam 2 Question #12
(Restated for clarity)

Which of the following sequences converge?

(1) $\left\{ a_n = \frac{2n}{3n+1} \right\} \to \frac{2}{3}$

(2) $\left\{ a_n = \cos(n\pi) \right\}$ diverges
$-1, +1, -1, +1, \ldots$

(3) $\left\{ a_n = n \sin\left(\frac{1}{n}\right) \right\}$? $\to 1$

A. (1), (2), & (3)
B. (2) & (3)
$\boxed{\text{C. (1) \& (3)}}$
D. (1) & (2)
E. (1)

$$\lim_{n\to\infty} \underbrace{n}_{\to +\infty} \underbrace{\sin\left(\tfrac{1}{n}\right)}_{\to 0} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right) \to 0}{\left(\frac{1}{n}\right) \to 0}$$

L'Hopitals

$$= \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cancel{\left(\frac{-1}{n^2}\right)}}{\cancel{\left(\frac{-1}{n^2}\right)}} = 1$$

I. Series:

Def: Given a sequence $\{a_1, a_2, a_3, \ldots\}$ the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an infinite series

The sequence of partial sums $\{S_n\}$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

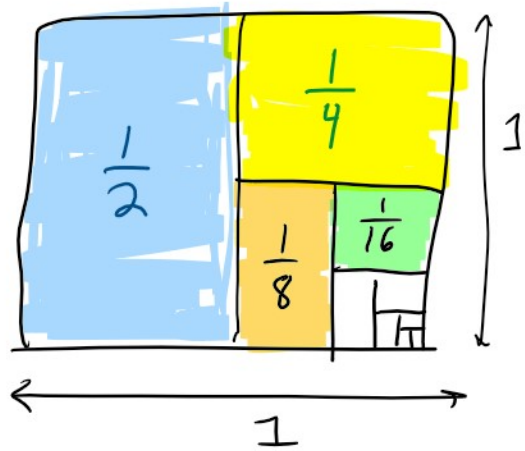
sum of the first n terms $\rightarrow$ $S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^{n} a_k$

If $\{S_n\}$ has a limit L, then the series <u>converges</u> to L

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L$$

If $\{S_n\}$ diverges, then the series also <u>diverges</u>

<u>Examples</u> (1): $a_k = \frac{1}{2^k}$



$a_1 = \frac{1}{2}$ $\quad S_1 = \frac{1}{2}$

$a_2 = \frac{1}{4}$ $\quad S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$a_3 = \frac{1}{8}$ $\quad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$S_n = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

$\lim_{n\to\infty} S_n$

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$$

<u>Example 2</u>: Consider the series

$0.9 + 0.09 + 0.009 + 0.0007 + \dots$

$$0.9 + 0.09 + 0.009 + 0.00\ldots$$

Here $a_k = \dfrac{9}{10^k} = 9(0.1)^k$

$$S_n = \sum_{k=1}^{n} 9(0.1)^k$$

$$S_1 = 0.9$$
$$S_2 = 0.9 + 0.09 = 0.99$$
$$S_3 = 0.9 + 0.09 + 0.009 = 0.999$$
$$\lim_{n\to\infty} S_n = 1$$

$$\sum_{k=1}^{\infty} 9(0.1)^k = 1$$

II. Geometric Series :

$$S_n = ar^0 + ar + ar^2 + \ldots + ar^{n-1}$$
$$= \sum_{k=0}^{n-1} ar^k$$

geometric sum

r - ratio

$$\sum_{k=0}^{\infty} ar^k \leftarrow \text{Geometric series}$$

$k=0$ ← starts at zero

Ex: $\boxed{0.9} + \boxed{0.09} + 0.009 + \ldots$

$\boxed{a} + \boxed{ar} + ar^2 + \ldots$

$a = 0.9$ $\quad ar = 0.09 = \underbrace{(0.9)}_{a}\underbrace{(0.1)}_{r}$

$$\sum_{k=0}^{\infty} 0.9\,(0.1)^k = 1$$

Given $$\sum_{k=0}^{\infty} ar^k = ?$$

$$S_n = ar^0 + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$$

$$rS_n = r\,(ar^0 + ar^1 + ar^2 + \dots + ar^{n-2} + ar^{n-1})$$

$$= ar^1 + ar^2 + \dots + ar^{n-2} + ar^{n-1} + ar^n$$

$$S_n - rS_n = ar^0 + \cancel{ar^1} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}}$$
$$- \{\cancel{ar^1} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}} + ar^n\}$$

$$S_n - rS_n = ar^0 - ar^n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$ * only if $r \neq 1$

$$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$r^n \xrightarrow{\text{if } |r|<1} 0$

$r^n \to +\infty$ if $|r|>1$

$$= \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } r=1,\ r=-1 \\ \text{diverges} & \text{if } |r|>1 \end{cases}$$

if $r=1$
$$\sum_{k=0}^{\infty} a(1)^k = a+a+a+a+\dots \to \text{diverges}$$

if $r=-1$ $$\sum_{k=0}^{\infty} a(-1)^k = a-a+a-a+\dots \to \text{diverges}$$

$$\boxed{\sum_{k=0}^{\infty} ar^k = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{otherwise} \end{cases}}$$

Example:

① $$\sum_{k=0}^{\infty} e^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{e}\right)^k$$

$a=1$
$r=\frac{1}{e}$
$|r|<1$

$$= \frac{a}{1-r} = \frac{(1)e}{\left(1-\frac{1}{e}\right)e} = \boxed{\frac{e}{e-1}}$$

② $$\sum_{k=2}^{\infty} 3(-0.75)^k$$

$k=2$ ← doesn't start @ zero

$$= \boxed{\underset{a}{3(-0.75)^2}} + \boxed{\underset{ar}{3(-0.75)^3}} + \underset{ar^2}{3(-0.75)^4} + \dots$$

$a = 3(-0.75)^2$

$$ar = 3(-0.75)^3 = \underbrace{3(-0.75)^2}_{a}\underbrace{(-0.75)}_{r}$$

$$= \sum_{k=0}^{\infty} \left[3(-0.75)^2\right](-0.75)^k$$

Geometric Series

$r = -0.75 \quad |r| < 1$

$$= \frac{a}{1-r} = \frac{3(-0.75)^2}{1-(-0.75)}$$

$$= \frac{3\left(-\frac{3}{4}\right)^2}{1+\frac{3}{4}} = \frac{3\cdot\frac{3^2}{4^{\cancel{2}}}}{\frac{7}{\cancel{4}}} = \frac{3^3}{7\cdot 4} = \boxed{\frac{27}{28}}$$

III. Telescoping Series

<u>Ex</u>: $a_k = \frac{1}{k(k+1)}$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$$

Instead rewrite using partial fractions

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

Telescoping series

$$S_n = 1 - \frac{1}{n+1}$$

$$\sum^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} 1 - \frac{1}{n+1} = 1$$