Lesson 21 - MA 166 LEC 200

10.4: Divergence Test

Announcements:
Exam 2 on Wed Mar 9 @ 6:30pm
Assigned seats
Review in Class on Wed

Warm Up: Fall 2015 Exam 2 Question #11

(this type of problem will be on exam 3)

11. $\sum_{k=1}^{\infty} \frac{2^k}{3^{k+1}} =$ ← Geometric?

A. 2/3

B. 1/2

C. 2/9

D. 3/4

E. 4/3

$$= \frac{2^1}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \cdots$$

$$= a + ar + ar^2 + \cdots$$

$a = \frac{2}{9}$ $\quad ar = \frac{2^2}{3^3} = \underbrace{\left(\frac{2}{9}\right)}_{a}\underbrace{\left(\frac{2}{3}\right)}_{r}$

$\sum_{k=0}^{\infty} \frac{2}{9}\left(\frac{2}{3}\right)^k \qquad |r| = \left|\frac{2}{3}\right| < 1$

$\boxed{A}$

$$= \frac{a}{1-r} = \frac{2/9}{1-\frac{2}{3}} = \frac{2/9}{1/3} = \frac{2}{9}\cdot\frac{3}{1} = \frac{2}{3}$$

Exam 2 Review on Wed:

Poll

Q: Does Exam 2 cover arc length?

A: NO

Exam 2 does cover: lessons 10-18

chap 8.1-8.5, 8.9 ← Improper Integrals

Integration Techniques $\int_1^{\infty} \frac{1}{x^2}dx$

Q: Exam 2 doesn't cover: √ volume of a

Q: Exam 2 doesn't cover:

- sequences + series (exam 3)
- centroid
- physical Applications
- trapezoid rule

volume of a solid
→ HW problems
↳ turns into an integration techniq. fair game

I. Divergence Test:

Q: When does a series diverge?

Theorem: (Divergence Test)

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$

OR

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges

NOTE: This only tells us when a series diverges

Ex: (1) $\sum_{k=1}^{\infty} \frac{k}{k+1}$ $\qquad a_k = \frac{k}{k+1}$

$$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1 \neq 0$$

the series diverges (by the Divergence Test)

Ex (2): $\sum_{k=1}^{\infty} \frac{1+3^k}{2^k}$ $\qquad a_k = \frac{1+3^k}{2^k}$

$$\lim \frac{1+3^k}{2^k} = \lim \underbrace{\frac{1}{2^k}}_{\to 0} + \underbrace{\left(\frac{3}{2}\right)^k}_{\to +\infty} = +\infty$$

$$\lim_{k\to\infty} \frac{1+3^k}{2^k} = \lim_{k\to\infty} \frac{1}{2^k} + \left(\frac{3}{2}\right)^k = +\infty$$

By the Divergence Test, the series diverges.

Ex (3): $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ $\qquad a_k = \frac{1}{k}$

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

Inconclusive

(Divergence Test can't be used to show a series converges)

(In fact, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverge)

Ex (4): $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ $\qquad a_k = \frac{1}{k^2}$

$$\lim_{k\to\infty} \frac{1}{k^2} = 0$$

Inconclusive

(In fact, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges)

Q: Why does the divergence test work

Want to show:

$$\left\{\begin{array}{ccc} A & \Longrightarrow & B \\ \displaystyle\lim_{k\to\infty} a_k \neq 0 & & \displaystyle\sum_{k=1}^{\infty} a_k \text{ diverges} \\ \text{not } A & \Longleftarrow & \text{not } B \end{array}\right.$$

not A $\Longleftarrow$ not B

$\lim_{k\to\infty} a_k = 0$ $\quad$ $\sum_{k=1}^{\infty} a_k$ converges

Assume $\sum_{k=1}^{\infty} a_k$ converges

the partial sums S_n have a finite limit

$\lim_{n\to\infty} S_n = L$ (where L is a finite number)

Recall $S_n = a_1 + a_2 + \cdots + a_n$

If $\lim_{n\to\infty} S_n = L \longrightarrow \sum_{k=1}^{\infty} a_k = L$

Want

$$\lim_{k\to\infty} a_k \overset{?}{=} \lim_{k\to\infty} \{ S_k - S_{k-1} \}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \{ (\cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_{k-1}} + a_k) - (\cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_{k-1}}) \}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \{ S_k - S_{k-1} \}$$

$$= \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1}$$

$$= L - L = 0$$

so $\lim_{k\to\infty} a_k = 0$.

so $\lim_{k\to\infty} a_k = 0$.

$$\boxed{\text{If } \sum_{k=1}^{\infty} a_k = L \quad \Rightarrow \quad \lim_{k\to\infty} a_k = 0}$$

$$\sum_{k=1}^{\infty} a_k \to L$$
$$S_n \to L$$
$$a_k \to 0$$

Review for Exam 2: Lessons 10 – 18

Lesson 10: Integration by Parts

$$\int u\,dv = u \cdot v - \int v\,du$$

Ex: $\int \ln x\,dx \to$ $u = \ln x$, $dv = dx$, $du = \frac{1}{x}dx$, $v = x$

$\hookrightarrow x\ln x - x + C$

Ex: $\int x\cos(x)\,dx \to$ $u = x$, $dv = \cos(x)\,dx$, $du = dx$, $V = \sin(x)$

$= x\sin(x) + \cos(x) + C$

Lesson 11: $\int \cos^m(x)\sin^n(x)\,dx = ?$

m	n	method
odd		$u = \sin(x)$ $du = \cos(x)\,dx$ $\cos^2(x) + \sin^2(x) = 1$

	odd	$u = \cos(x)$ $du = -\sin(x)\,dx$
even	even	Half angle formulas $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$ $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

Lesson 12: $\int \tan^m(x)\sec^n(x)\,dx = ?$

m	n	method
	even	$u = \tan(x)$ $du = \sec^2(x)\,dx$ $\tan^2(x)+1 = \sec^2(x)$
odd	odd	$u = \sec(x)$ $du = \tan(x)\sec(x)\,dx$
even	odd	Integration by parts $\tan^2(x)+1 = \sec^2(x)$

Lessons 13-15: Trig Substitution

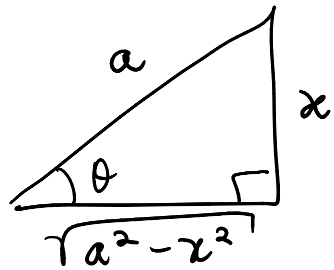
$\boxed{1}$ $\sqrt{a^2 - x^2}$ Let $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

$\downarrow$

$$\sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

Reference Triangle



$x = a\sin\theta$

$\sin\theta = \frac{x}{a}$

$\boxed{2}$ $\sqrt{a^2 + x^2}$

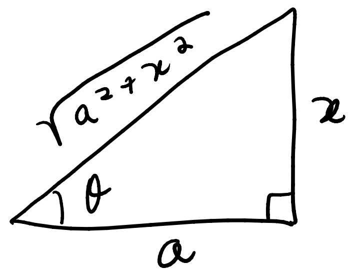
Let $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$\downarrow$

$\sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$

Reference Triangle



$x = a\tan\theta$

$\tan\theta = \frac{x}{a}$

$\boxed{3}$ $\sqrt{x^2 - a^2}$

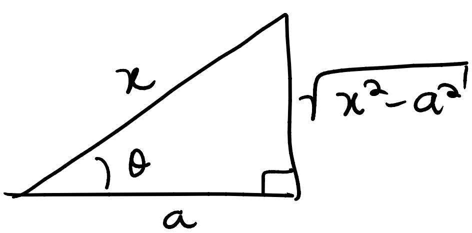
Let $x = a\sec\theta$

$dx = a\tan\theta\sec\theta\, d\theta$

$\downarrow$

$\sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$

Reference Triangle



$x = a\sec\theta$

$\sec\theta = \frac{x}{a}$

☆ May need to complete the square first

i.e. $\sqrt{x^2 + bx + c} \xrightarrow{\text{rewrite}} \sqrt{(x-d)^2 \pm a^2}$

$$\int \frac{x^2 + x + 2}{x^2+2x+2} = \int \frac{x^2+2x+2-x}{x^2+2x+2}$$

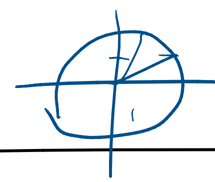
$$= \int 1 - \frac{x}{x^2+2x+2}\, dx \qquad \frac{x}{x^2+2x+2} = \frac{Ax+B}{x^2+2x+2}$$

$$= \int 1 - \frac{x}{(x+1)^2+1}\, dx \qquad \begin{aligned} u &= x+1 \\ du &= dx \end{aligned}$$

$$= \int 1\,dx - \int \frac{u-1}{u^2+1}\, du$$

$$\int dx - \int \frac{u}{u^2+1} du + \int \frac{1}{u^2+1} \leftarrow \tan^{-1}(u)$$

$$\begin{aligned} v &= u^2+1 \\ dv &= 2u\,du \end{aligned}$$



8.4 #11

$$\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\, dx \qquad \begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \end{aligned}$$

$\sqrt{3} = 2\sin\theta$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}$

$1 = 2\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$$\int_{\pi/6}^{\pi/3} \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}} = \int^{4} \frac{\cancel{8}\sin^2\theta \cancel{\cos\theta}\, d\theta}{\cancel{2}\cancel{\cos\theta}}$$

$$= 4\int \sin^2\theta\, d\theta = 4\int_{\pi/6}^{\pi/3} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= \frac{4}{2}\left[\theta - \sin(2\theta)\right]$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$\rightarrow$ Reference Triangle

Find p $\int_1^\infty \frac{1}{x^{1-p}}\,dx$ converges

$$= \lim_{b\to\infty} \int_1^b x^{p-1}\,dx = \lim_{b\to\infty} \left[\frac{x^p}{p}\right]_1^b$$ ← assume $p \neq 0$

$$= \lim_{b\to\infty} \left[\frac{b^p}{p} - \frac{1^p}{p}\right] = \frac{1}{p}$$

if $p=-2$

$\lim_{b\to\infty} \frac{1}{b^2} = 0$

want $\lim_{b\to\infty} b^p = \infty$ if $p>0$

$\lim_{b\to\infty} \frac{1}{b^p} = 0$ if $p<0$

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$$\int_1^\infty \frac{dx}{x^{1-p}} = \int_1^\infty x^{p-1}\,dx = \lim_{b\to\infty}\int_1^b x^{p-1}\,dx$$

$$= \begin{cases} \lim_{b\to\infty} \int_1^b \frac{dx}{x} & p=0 \\ \lim_{b\to\infty} \int_1^b x^{p-1}\,dx & p\neq 0 \end{cases}$$

$$= \begin{cases} \lim_{b\to\infty} \left[\ln x\right]_1^b & p=0 \\ \lim_{b\to\infty} \left[\frac{x^p}{p}\right]_1^b & p\neq 0 \end{cases}$$

$$= \begin{cases} \lim_{b\to\infty} \left[\ln b - \ln 1\right] & p=0 \\ \lim_{b\to\infty} \frac{1}{p}\left[b^p - 1^p\right] & p\neq 0 \end{cases}$$

$$= \begin{cases} \text{diverges} & p=0 \\ \text{diverges} & p>0 \end{cases}$$

$$- \begin{cases} \text{diverges} & p > 0 \\ \text{converges} & p < 0 \end{cases}$$

* I think there is a typo in the problem, it should read

$$\int_1^\infty x^{1-p}\,dx$$

Then $\lim\limits_{b\to\infty} \int_1^b x^{1-p}\,dx$

$$= \begin{cases} \lim\limits_{b\to\infty} \int_1^b \frac{dx}{x} & p = 2 \\ \lim\limits_{b\to\infty} \int_1^b x^{1-p} & p \neq 2 \end{cases}$$

$$= \begin{cases} \lim\limits_{b\to\infty} \left[\ln x\right]_1^b & p = 2 \\ \lim\limits_{b\to\infty} \left[\frac{x^{1-p+1}}{2-p}\right]_1^b & p \neq 2 \end{cases}$$

$$= \begin{cases} \lim\limits_{b\to\infty} \left[\ln(b) - \ln(1)\right] & p = 2 \\ \lim\limits_{b\to\infty} \left[\frac{b^{2-p}}{2-p} - \frac{1^{2-p}}{2-p}\right] & p \neq 2 \end{cases}$$

$$= \begin{cases} \text{diverges} & \text{if} & p = 2 \\ \text{diverges} & \text{if} & p < 2 \\ \text{converges} & \text{if} & p > 2 \end{cases}$$