10.4: Integral Test and p-Series

Announcements:
- Exam 2 scores released Wed

Warm Up: According to the Divergence Test, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ ← diverges!

(a) Converges (b) Diverges (c) Inconclusive

Divergence Test: ⟹

If $\lim_{k\to\infty} a_k \neq 0$ then $\sum_{k=1}^{\infty} a_k$ diverges

NOT: $\lim_{k\to\infty} a_k = 0 \not\Rightarrow \sum a_k$ converges

I. Harmonic Series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Q: Does this series converge?

partial sums $S_n = \sum_{k=1}^{n} \frac{1}{k}$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

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$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

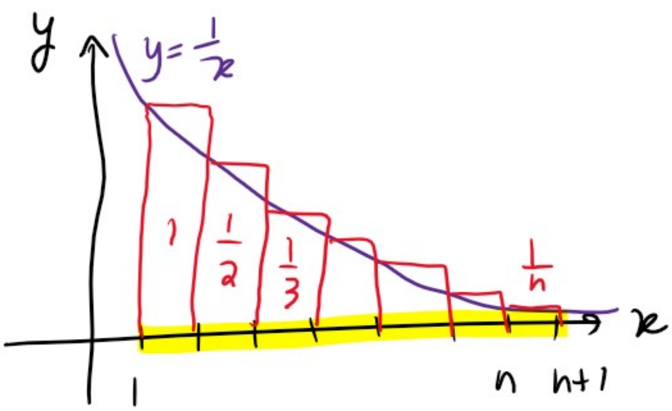
$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$

no obvious sequence

nth partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$\rightarrow$ Riemann sum of $y = \frac{1}{x}$



Riemann sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

Riemann sum

$$\sum_{k=1}^{n} \frac{1}{k} > \int_1^{n+1} \frac{1}{x}\,dx$$

$$= \Big[\ln|x| \Big]_1^{n+1}$$

$$= \ln(n+1) - \cancel{\ln(1)}^{\,0} = \ln(n+1)$$

$$S_n > \ln(n+1)$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \lim_{n\to\infty} S_n > \lim_{n\to\infty} \ln(n+1) = +\infty$$

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges because $\int_1^{\infty} \frac{1}{x}\,dx$ diverges

II. Integral Test:

$\{a_k\}_{k=1}^{\infty}$ is a sequence

$f(x)$ is
- continuous
- positive
- decreasing

on $[1, \infty)$

and $a_k = f(k)$ $\quad k = 1, 2, 3, \ldots$

Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$

Either:
1) both converge
2) both diverge

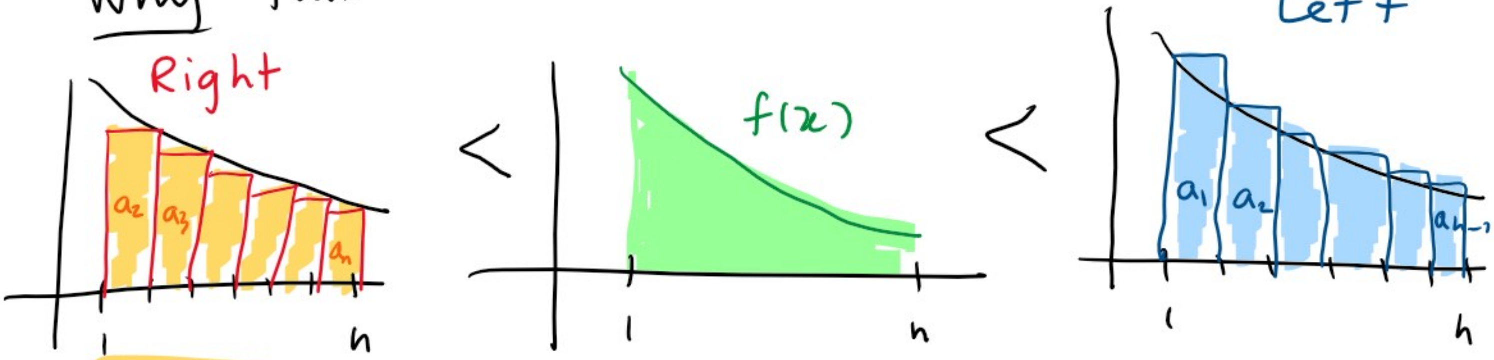
NOTE: If both converge

$$\sum_{k=1}^{\infty} a_k \neq \int_1^{\infty} f(x)\,dx$$

Why this works — Riemann sums

Left

Why this works — Riemann



$$\boxed{\sum_{k=2}^{n} a_k} \quad < \quad \int_1^n f(x)\,dx \quad < \quad \sum_{k=1}^{n-1} a_k$$

Q: $\displaystyle\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$

$$S_n = \sum_{k=1}^{n} a_k = a_1 + \boxed{\sum_{k=2}^{n} a_k}$$

$$\boxed{< a_1 + \int_1^n f(x)\,dx}$$

If $\int_1^\infty f(x)\,dx$ converges, then

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n < \lim_{n\to\infty} a_1 + \int_1^n f(x)\,dx$$

$$= a_1 + \int_1^\infty f(x)\,dx < +\infty$$

If $\int_1^\infty f(x)\,dx \Rightarrow \displaystyle\sum_{k=1}^{\infty} a_k$

Integral Test

Ex: ① $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ Integral Test

$a_k = \frac{k}{k^2+1}$ $f(x) = \frac{x}{x^2+1}$

✓ continuous on $[1,\infty)$
✓ positive on $[1,\infty)$
✓ decreasing on $[1,\infty)$

Check 1st derivative

$f'(x) < 0 \Longrightarrow$ decreasing

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{if} \quad x > 1$$

Integral Test:

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{x}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{x\,dx}{x^2+1}$$

$u = x^2+1$
$du = 2x\,dx$
$x\,dx = \frac{du}{2}$

$$= \lim_{b\to\infty} \frac{1}{2}\int_2^{b^2+1} \frac{du}{u}$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln(u)\Big]_2^{b^2+1}$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln(b^2+1) - \ln(2)\Big] = +\infty$$

Integral Diverges

$\Longrightarrow$ Series also diverges.

$\Rightarrow$ Series also diverges.

(when $p=1$ $\rightarrow$ harmonic series)

III. p-Series:

The p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p>1$ and diverges for $p\le 1$

NOTE: Can prove using the Integral Test and the Divergence Test.

Ex: (1) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/4}}$

Here $p = 3/4$, $p \le 1 \rightarrow$ diverges

(2) $\displaystyle\sum_{k=4}^{\infty} \frac{1}{(k-1)^2}$

Look at first few terms

$$= \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2}$$

$$= \sum_{k=3}^{\infty} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$p=2$

$p>1$

$\uparrow$ converges

$\uparrow$ looks like

$\uparrow$

looks like p-series

$\leftarrow$ converges

$$\sum_{k=3}^{\infty} \frac{1}{k^2} \text{ converges} \rightarrow \sum_{k=4}^{\infty} \frac{1}{(k-1)^2} \text{ converges}$$

IV. Estimating the value of a Series:

Look at the remainder of a series

$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = a_{n+1} + a_{n+2} + \cdots$$

We find that

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$$

If the series converges to S

$$S = \sum_{k=1}^{\infty} a_k = S_n + R_n$$

$$\underbrace{S_n + \int_{n+1}^{\infty} f(x)\,dx}_{\substack{L_n \\ \text{lower bound}}} < \overset{S}{\overset{\shortparallel}{S_n + R_n}} < \underbrace{S_n + \int_{n}^{\infty} f(x)\,dx}_{\substack{U_n \\ \text{upper bound}}}$$

Ex: How many terms of the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation within 10^{-3} of the value of the series S?

want n st. $|S - S_n| < 10^{-3}$

$$R_n < 10^{-3}$$

$$R_n < \int_n^\infty f(x)dx = \int_n^\infty \frac{1}{x^2} dx$$

$$= \lim_{b\to\infty} \int_n^b \frac{1}{x^2} dx = \lim_{b\to\infty} \left[\frac{x^{-1}}{-1} \right]_n^b$$

$$= \lim_{b\to\infty} \left[\underbrace{-\frac{1}{b}}_{\to 0} - \left(\frac{-1}{n}\right) \right] = \frac{1}{n}$$

Want

$$R_n < 10^{-3}$$

$$\frac{1}{n} < 10^{-3}$$

$$10^3 < n$$

Need at least $n = 1001$ terms to be sure $R_n < 10^{-3}$