Lesson 22 - MA 166 LEC 286

Announcements:
- Exam 2 scores released Wed

10.4: Integral Test and p-Series

Warm Up: According to the Divergence Test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$

(a) Converges (b) Diverges $\boxed{\text{(c) Inconclusive}}$

Divergence Test

If $\displaystyle\lim_{k\to\infty} a_k \neq 0 \Longrightarrow \sum_{k=1}^{\infty} a_k$ diverges

$\displaystyle\lim_{k\to\infty} a_k = 0 \;\cancel{\Longrightarrow}\; \sum_{k=1}^{\infty} a_k$ converges

I. Harmonic Series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Q: Does this converge?

partial sums $\displaystyle S_n = \sum_{k=1}^{n} \frac{1}{k}$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

no obvious

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

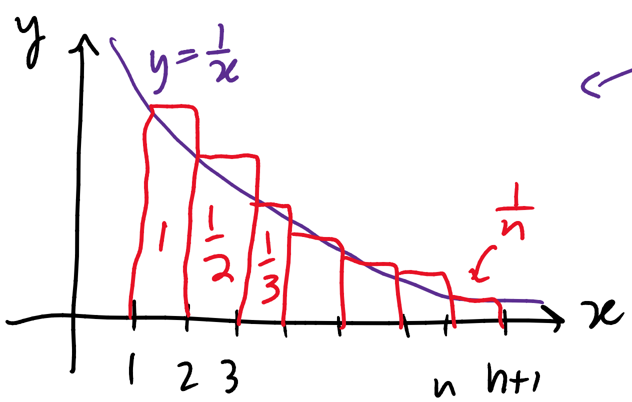
$$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

no obvious sequence

Instead $S_n = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$\rightarrow$ Riemann sum of $y = \frac{1}{x}$



Riemann sum over estimates the area under $\frac{1}{x}$

Riemann sum

$$\sum_{k=1}^{n} \frac{1}{k} = S_n$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k} > \int_1^{n+1} \frac{1}{x}\,dx$$

$$= \Big[\ln|x|\Big]_1^{n+1}$$

$$= \ln(n+1) - \cancel{\ln(1)}^{\,0} = \ln(n+1)$$

So $S_n > \ln(n+1)$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \lim_{n\to\infty} S_n > \lim_{n\to\infty} \ln(n+1) = +\infty$$

$$\sum_{k=1} \frac{1}{k} = \lim_{n\to\infty} S_n > \lim_{n\to\infty} \ldots$$

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges because } \int_1^{\infty} \frac{1}{x}\,dx \text{ diverges}$$

II. Integral Test:

$\{a_k\}_{k=1}^{\infty}$ is a sequence

$f(x)$ is
- continuous
- positive
- decreasing

on $[1, \infty)$

and $a_k = f(k)$ $\quad k = 1, 2, 3, \ldots$

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

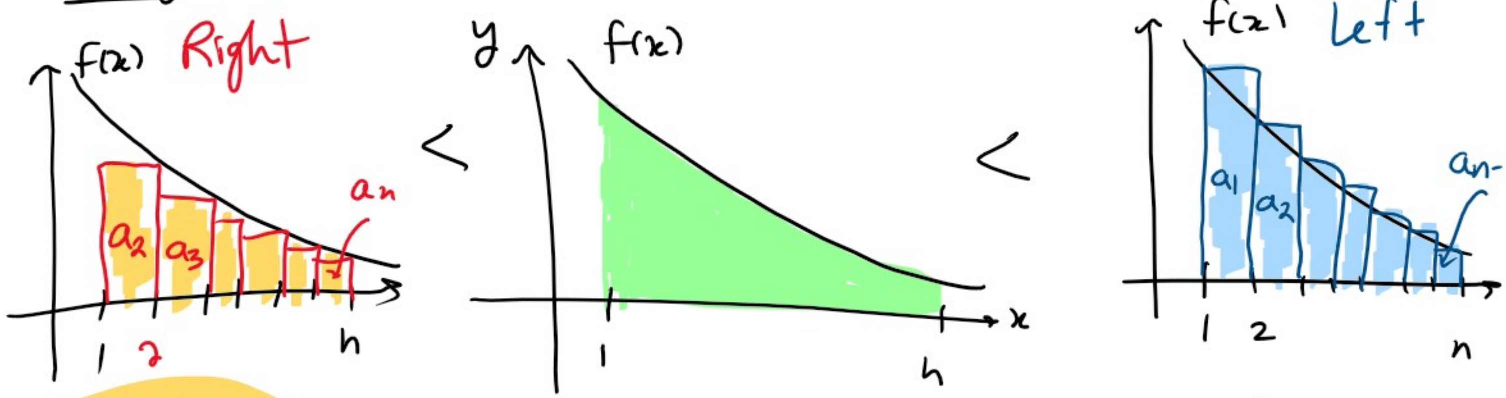
Then either (1) both converge
(2) both diverge

NOTE: If the integral converges

$$\sum_{k=1}^{\infty} a_k \neq \int_1^{\infty} f(x)\,dx$$

Why this works — Riemann Sums

$f(x)$ Right $\quad y \uparrow f(x) \quad$ $f(x)$ Left



$$\sum_{k=2}^{n} a_k \boxed{<} \int_1^n f(x)\,dx < \sum_{k=1}^{n-1} a_k$$

Assumed $\int_1^\infty f(x)\,dx$ converges

want to show: WTS $\lim_{n\to\infty} S_n$ converges

$$S_n = \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k$$

$$\boxed{<} a_1 + \int_1^n f(x)\,dx$$

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n < \lim_{n\to\infty} a_1 + \int_1^n f(x)\,dx$$

$$= a_1 + \int_1^\infty f(x)\,dx < \infty$$

converges

converges

<u>Example</u>: $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

Example: $\displaystyle\sum_{k=1} \frac{k}{k^2+1}$

$a_k = \dfrac{k}{k^2+1}$ $\qquad f(x) = \dfrac{x}{x^2+1}$

✓ continuous on $[1,\infty)$
✓ positive
? decreasing

How to show is decreasing ✓
$f'(x) < 0$ $\qquad f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ if $x > 1$

Apply the integral test:

$$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx \qquad u = x^2+1$$

$$= \lim_{b\to\infty} \frac{1}{2}\int_2^{b^2+1} \frac{du}{u} = \lim_{b\to\infty}\left[\ln(b^2+1) - \ln(2)\right] = +\infty$$

Integral Diverges
$\rightarrow$ Series Diverges

III. p-series

If $p=1$ $\longrightarrow$ Harmonic Series

The p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$, diverges if $p \leq 1$

NOTE: Can prove this using Integral Test + with the Divergence Test

$\sum^\infty \quad \frac{1}{\ } \qquad \sum^\infty \frac{1}{\ } \qquad p = \frac{3}{\ } < 1$

Ex: (1) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/4}}$ $\quad p = \frac{3}{4} < 1$

diverge

(2) $\sum_{k=4}^{\infty} \frac{1}{(k-1)^2}$ write out 1st few terms

$= \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$

$= \sum_{k=3}^{\infty} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2}$

$\left(\frac{1}{3^2} + \frac{1}{4^2} + \cdots \right)$ $\quad \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right.$

Here $p = 2 > 1$

this series converges $\Longleftarrow$ By the p-series test this converges

IV. Estimating the value of a series:

Look at the remainder of a series

$$R_n = \underbrace{\sum_{k=1}^{\infty} a_k}_{\text{full series}} - \underbrace{\sum_{k=1}^{n} a_k}_{S_n} = a_{n+1} + a_{n+2} + \cdots$$

We find that

$$\int^{\infty} f(x)\,dx < R_n < \int^{\infty} f(x)\,dx$$

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$$

If series converges,

$$S = \sum_{k=1}^{\infty} a_k = S_n + R_n$$

$$\underbrace{S_n + \int_{n+1}^{\infty} f(x)\,dx}_{\substack{L_n \\ \text{lower bound}}} < S < \underbrace{S_n + \int_{n}^{\infty} f(x)\,dx}_{\substack{U_n \\ \text{upper bound}}}$$

Ex: How many terms of the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation within 10^{-3} of the value of the series S?

$$R_n = |S - S_n| < 10^{-3}$$

$$R_n < \int_{n}^{\infty} f(x)\,dx = \int_{n}^{\infty} \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \int_{n}^{b} \frac{dx}{x^2} = \lim_{b\to\infty} \left[\frac{x^{-1}}{-1}\right]_{n}^{b}$$

$$= \lim_{b\to\infty} \left[\underbrace{-\frac{1}{b}}_{\to 0} - \left(\frac{-1}{n}\right)\right] = \frac{1}{n}$$

$$\frac{1}{n+1} < R_n < 10^{-3}$$

$$\frac{1}{n+1} <_{ss} R_n < 10^{-3}$$

$$\frac{1}{n} < 10^{-3}$$

$$\boxed{10^3 < n+1} \rightarrow n > 999$$

$$R_n > \int_{n+1}^{\infty} f(x)\,dx = \frac{1}{n+1}$$