Lesson 23 - MA 166 LEC 200

10.5: Comparison Test & Limit Comparison Test

Announcements:

Exam 2 scores released

☆ Warm-Up:

Use the Integral Test to determine if the following series converges: $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^k}$

$\boxed{\text{(a) converges}}$ (b) diverges

$$\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty} \int_1^b x e^{-x}\,dx \quad \to \text{IBP} \quad u = x \quad dv = e^{-x}dx$$

$$= \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\underbrace{-be^{-b}}_{\to 0 \text{ L'Hopital's}} - \underbrace{e^{-b}}_{\to 0} + e^{-1} + e^{-1}\right] = \frac{2}{e}$$

integral converges

→ series converges.

I. Series: Review

Name	Series	Convergence?
Geometric Series	$\displaystyle\sum_{k=0}^{\infty} ar^k$	$= \dfrac{a}{1-r}$ if $\lvert r\rvert<1$
Harmonic	$\displaystyle\sum^{\infty} \frac{1}{\ }$	Diverges

Harmonic Series	$\sum_{k=1}^{\infty} \frac{1}{k}$	Diverges
p-Series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	converges if $p > 1$
Telescoping series	$\sum_{k=1}^{\infty} f(k) - f(k+1)$	Converges to $f(1)$

II. Comparison Test:

(Direct) Comparison Test:

Let $\sum a_k$ and $\sum b_k$ be series with positive terms, then

(1) If $a_k \leq b_k$ and $\sum b_k$ converges

$\Rightarrow \sum a_k$ converges

(2) If $a_k \geq b_k$ and $\sum b_k$ diverges

$\Rightarrow \sum a_k$ diverges

Ex: $\sum_{k=1}^{\infty} \frac{1}{k^2+10}$ ← unknown series

$a_k = \frac{1}{k^2+10}$ Want to compare to

Try: $b_k = \frac{1}{k^2}$ ← p-series $p = 2 > 1$ converges

$$a_k = \frac{1}{k^2+10} \leq \frac{1}{k^2} = b_k \to \text{converges}$$

$k^2+10 \geq k^2$

$\frac{1}{k^2} \geq \frac{1}{k^2+10}$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2+10}$ also converges by the Comparison Test.

<u>Ex</u> (2) $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k-3}} \to a_k = \frac{1}{\sqrt{k-3}}$

Try $b_k = \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$

$\sqrt{k-3} \leq \sqrt{k}$

$\frac{1}{\sqrt{k}} \leq \frac{1}{\sqrt{k-3}}$

$$a_k = \frac{1}{\sqrt{k-3}} \geq \frac{1}{\sqrt{k}} = b_k$$

$p = \frac{1}{2} < 1$ diverges

$\Downarrow$ $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k-3}}$ also diverges by the Comparison Test

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III. Limit Comparison Test:

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L$$

Then: ... then either

Then:

(1) If $0 < L < \infty$ then either both series converge or both diverge

(2) If $L = 0$ and $\sum b_k$ converges then $\sum a_k$ also converges

(3) If $L = \infty$ and $\sum b_k$ diverges then $\sum a_k$ also diverges

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Ex: (1) $\displaystyle\sum_{k=1}^{\infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$

Let: $a_k = \dfrac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$

$b_k = ?$ $\rightarrow$ look at the leading terms of a_k

$a_k \sim \dfrac{k^4}{k^6} = \dfrac{1}{k^2}$

Let $b_k = \dfrac{1}{k^2}$

Apply L.C.T

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\left(\dfrac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}\right)}{\left(\dfrac{1}{k^2}\right)}$$

$= 5k^6 - 2k^4 + 3k^2 \left(\frac{1}{k^6}\right)$

$$= \lim_{k\to\infty} \frac{5k^6 - 2k^4 + 3k^3}{2k^6 - k + 5} \frac{\left(\frac{1}{k^6}\right)}{\left(\frac{1}{k^6}\right)}$$

$$= \lim_{k\to\infty} \frac{5 - 2\left(\frac{1}{k^2}\right) + 3\left(\frac{1}{k^4}\right)}{2 - \frac{1}{k^5} + 5\left(\frac{1}{k^6}\right)} = \frac{5}{2}$$

$\sum a_k$ and $\sum b_k$ both converge or diverge

$\sum b_k = \sum \frac{1}{k^2}$ p-series $p = 2 > 1 \rightarrow$ converges

$\sum a_k$ also converges.

Q: Why does this work?

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} \longleftarrow \text{if } \sum b_k \text{ converges}$$

If $0 < L < \infty \rightarrow \sum a_k$ has the same growth rate as $L \cdot \sum b_k$

Conversely

$$\infty = L = \lim_{k\to\infty} \frac{a_k}{b_k} \longleftarrow \text{if } \sum b_k \text{ diverges}$$

$\sum a_k$ grows faster $\sum b_k$ which diverges

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<u>Ex: (2)</u> $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ $\qquad a_k = \frac{\ln(k)}{k^2}$

#1 <u>Try</u>: $b_k = \frac{1}{k^2}$ $\longrightarrow$ p-series $\sum b_k$
$p=2 \to$ converges

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\left(\frac{\ln(k)}{k^2}\right)}{\left(\frac{1}{k^2}\right)} = \lim_{k\to\infty} \ln(k)$$
$$= \infty$$

$\Rightarrow\Leftarrow$

L.C.T is inconclusive

a_k decays slower $\frac{1}{k^2}$

<u>Try #2</u>: $b_k = \frac{1}{k}$ $\leftarrow$ Harmonic series
Diverges

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\left(\frac{\ln(k)}{k^2}\right)}{\left(\frac{1}{k}\right)} = \lim_{k\to\infty} \frac{\ln(k)}{k} \begin{matrix} -\infty \\ -\infty \end{matrix}$$

L'Hopital's $= \lim\limits_{k\to\infty} \frac{\left(\frac{1}{k}\right)}{1} = 0$

$\Rightarrow\Leftarrow$

L.C.T. is inconclusive

$\sum a_k$ decays faster $\frac{1}{k}$

Try #3 : $b_k = \frac{1}{k^{3/2}}$ $\qquad \frac{1}{k} < \cdot < \frac{1}{k^2}$

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\left(\frac{\ln(k)}{k^2}\right)}{\left(\frac{1}{k^{3/2}}\right)}$$

$$= \lim_{k\to\infty} \frac{\ln(k)}{k^{1/2}} \quad - \frac{\infty}{\infty} \quad \text{L'Hopital's Rule}$$

$$= \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)}{\frac{1}{2}k^{-1/2}} = \lim_{k\to\infty} \frac{2}{\sqrt{k}} = 0$$

$L = 0$ $\qquad \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \rightarrow$ p-series
$p = \frac{3}{2} > 1 \rightarrow$ converges

$\rightarrow$ By L.C.T. $\sum_{k=1}^{\infty} a_k$ also converges.

NOTE: We can also show that $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ converges using the Integral Test