Lesson 23 - MA 166 LEC 286

10.5: Comparison Test & Limit Comparison Test

Announcements:

Exam 2 scores released
→ REC Brightspace Gradebook

☆ Warm-Up:

Use the Integral Test to determine if the following series converges: $\sum_{k=1}^{\infty} \frac{k}{e^k}$

$\boxed{\text{(a) converges}}$ (b) diverges

Integral Test

IBP

$$\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty} \int_1^b x e^{-x}\,dx \qquad u = x \quad dv = e^{-x}dx$$

$$= \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty} \left(\underbrace{\cancel{-be^{-b}}}_{\infty\cdot 0} \xrightarrow{} 0 - \underbrace{e^{-b}}_{\to 0} + 1\cdot e^{-1} + e^{-1}\right]$$

L'Hopitals

$$\lim_{b\to\infty} \frac{-b}{e^b} \left(\frac{\infty}{\infty}\right) = \lim_{b\to\infty} \frac{-1}{e^b} = 0$$

$$\int_1^{\infty} \frac{x}{e^x}\,dx = \frac{2}{e} \implies \sum_{k=1}^{\infty} \frac{k}{e^k} \text{ converges}$$

I. Series: Review

Name	Series	Convergence?
Geometric Series	$\sum_{k=0}^{\infty} ar^k$	$= \frac{a}{1-r}$ if $\lvert r\rvert<1$
Harmonic Series	$\sum_{k=1}^{\infty} \frac{1}{k}$	Diverges
p-Series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Converges if $p>1$
Telescoping series	$\sum_{k=1}^{\infty} f(k)-f(k+1)$	converges to $f(1)$

II. Comparison Test:

(Direct) Comparison Test:

Let $\sum a_k$ and $\sum b_k$ be series with positive terms, then

(1) If $a_k \le b_k$ and $\sum b_k$ converges

$\Rightarrow$ $\sum a_k$ also converges

(2) If $a_k \ge b_k$ and $\sum b_k$ diverges

$\Rightarrow$ $\sum a_k$ also diverges

$\Rightarrow \sum a_k$ also diverges

Ex (1) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+10}$ $\leftarrow$ unknown series

let $a_k = \dfrac{1}{k^3+10}$

want to compare to something we know

$b_k = \dfrac{1}{k^3}$ p-series $p=3>1$ $\sum b_k$ converges

$$\frac{1}{k^3+10} = a_k \leq b_k = \frac{1}{k^3}$$

$$k^3+10 \geq k^3$$

$$\frac{1}{k^3} \geq \frac{1}{k^3+10}$$

$a_k \leq b_k$

$\sum b_k$ converges

$\Rightarrow \sum a_k$ also converges by the Comparison Test

Ex (2): $\displaystyle\sum_{k=4}^{\infty} \frac{1}{\sqrt{k-3}}$ $\quad a_k = \dfrac{1}{\sqrt{k-3}}$

$b_k = \dfrac{1}{\sqrt{k}} = \dfrac{1}{k^{1/2}}$ p-series $p=\frac{1}{2}$ $\rightarrow$ diverges

$b_k = \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$ p series, $p = \frac{1}{2}$ $\rightarrow$ diverges

$$\frac{1}{\sqrt{k-3}} = a_k \geq b_k = \frac{1}{\sqrt{k}}$$

$$\sqrt{k-3} \leq \sqrt{k}$$

$$\frac{1}{\sqrt{k}} \leq \frac{1}{\sqrt{k-3}}$$

$a_k \geq b_k$ and $\sum b_k$ diverges

$\Rightarrow$ $\sum a_k$ also diverges by the comparison Test.

NOTE: Comparison Test

Inequality	$\sum b_k$		$\sum a_k$
$a_k \leq b_k$	converges	$\Rightarrow$	converges
	diverges	$\Rightarrow$	Inconclusive
$a_k \geq b_k$	diverges	$\Rightarrow$	diverge
	converge	$\Rightarrow$	Inconclusive

III. Limit Comparison Test:

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L$$

L	$\sum b_k$		$\sum a_k$
$0 < L < \infty$	converges	$\Rightarrow$	converges
	diverges	$\Rightarrow$	diverges
$L = 0$	converges	$\Rightarrow$	converges
	diverges	$\Rightarrow$	Inconclusive
$L = \infty$	diverges	$\Rightarrow$	diverges
	converges	$\Rightarrow$	Inconclusive

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$

$$a_k = \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5} \qquad b_k = \ ?$$

look at the leading terms of a_k

$2k^6 - k + 5$ of a_k

$$b_k = \frac{1}{k^2} \quad \text{p-series } p=2 \text{ converges}$$

$$a_k \sim \frac{k^4}{k^6} = \frac{1}{k^2}$$

L.C.T:

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\left(\frac{5k^4-2k^2+3}{2k^6-k+5}\right)}{\left(\frac{1}{k^2}\right)}$$

$$= \lim_{k\to\infty} \frac{5k^6-2k^4+3k^2}{2k^6-k+5} \frac{\left(\frac{1}{k^6}\right)}{\left(\frac{1}{k^6}\right)}$$

$$= \lim_{k\to\infty} \frac{5-2\left(\frac{1}{k^2}\right)+3\left(\frac{1}{k^4}\right)}{2-\left(\frac{1}{k^5}\right)+5\left(\frac{1}{k^6}\right)} = \frac{5}{2}$$

$L = \frac{5}{2}$ $\quad \sum b_k$ converges $\Rightarrow$ $\sum a_k$ also converges.

Why this works

$$L = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$0 < L < \infty$ $\quad$ " $\sum a_k \sim L \cdot \sum b_k$ "

$L = 0$ $\quad$ " $\sum a_k << \sum b_k$ "

$L = \infty$ "$\sum a_k >> \sum b_k$"

<u>Ex</u>: $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ $a_k = \frac{\ln(k)}{k^2}$

<u>Try #1</u> $b_k = \frac{1}{k^2}$ p-series $p=2$ $\rightarrow$ converges

$$L = \lim_{k\to\infty} \frac{\left(\frac{\ln(k)}{k^2}\right)}{\left(\frac{1}{k^2}\right)} = \lim_{k\to\infty} \ln(k) = \infty$$

$L = \infty$ $\sum b_k$ converges $\Rightarrow$ Inconclusive

$$\frac{1}{k^2} < a_k$$

because $\frac{1}{k^2} < \frac{1}{k}$

<u>Try #2</u>: $b_k = \frac{1}{k}$ Harmonic series $\rightarrow$ Diverges

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\left(\frac{\ln(k)}{k^2}\right)}{\left(\frac{1}{k}\right)} = \lim_{k\to\infty} \frac{\ln(k)}{k} \quad \frac{\infty}{\infty}$$

L'Hopitals $= \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)}{1} = 0$

$L = 0$ $\sum b_k$ diverges $\Rightarrow$ Inconclusive

$L = 0$ $\quad \sum b_k$ diverges $\Rightarrow$ Inconclusive

$$\frac{1}{k^2} < a_k < \frac{1}{k}$$

Try #3 : $b_k = \dfrac{1}{k^{3/2}}$ p-series $p = \frac{3}{2}$ converges

$$L = \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\left(\frac{\ln(k)}{k^2}\right)}{\left(\frac{1}{k^{3/2}}\right)}$$

$$= \lim_{k \to \infty} \frac{\ln(k)}{\sqrt{k}} \quad \frac{\infty}{\infty} \to \text{L'Hopital's}$$

$$= \lim_{k \to \infty} \frac{\left(\frac{1}{k}\right)}{\frac{1}{2}k^{-1/2}} = \lim_{k \to \infty} \frac{2}{\sqrt{k}} = 0$$

$L = 0$ $\quad \sum b_k$ converges $\Rightarrow$ $\sum a_k$ also converges by the Limit Comparison Test.

NOTE: $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ can also use the Integral Test to show convergence.