10.6: Alternating Series Test

Warm Up: Use the Comparison Test to determine if the following series converges:

$$\sum_{k=1}^{\infty} \left(\frac{4k}{2k-3}\right)^k$$

4k (not 45 like on HotSeat)

(a) converges

$\boxed{\text{(b) diverges}}$

$$a_k = \left(\frac{2}{1-\frac{3}{2k}}\right)^k$$

$b_k = 2^k$ → Geometric series $r = 2 > 1$ diverges

$$a_k = \left(\frac{2}{1-\frac{3}{2k}}\right)^k \geq 2^k = b_k \longrightarrow \text{diverges}$$

↑ also diverges.

I. Alternating Series:

So far $\sum_{k=1}^{\infty} a_k$ assuming $a_k > 0$

An <u>alternating series</u> is an infinite series where terms alternate positive and negative

Ex: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ → 1st term post

↑ Alternating Harmonic Series

Ex: $\sum \frac{(-1)^k}{\ldots} = \frac{-1}{2} + \frac{1}{5} + \frac{-1}{10} + \cdots$ → 1st term neg

Ex: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1} = \frac{-1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots$

NOTE:

$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, $a_k > 0$

$\Rightarrow$ starts w/ pos term

$\sum_{k=1}^{\infty} (-1)^{k} a_k$, $a_k > 0$

$\Rightarrow$ starts w/ neg term

the math is the same for both types

Q: When does an alternating series converge?

Alternating Series Test:

$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, where $a_k > 0$

converges provided:

(1) terms are non-increasing in magnitude

$a_k \geq a_{k+1} > 0$ for $k > N$

(2) $\lim_{k\to\infty} a_k = 0$

Ex: (1) Alternating Harmonic Series

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ $\qquad a_k = \frac{1}{k}$

A.S.T (1) nonincreasing ✓

$a_k = \frac{1}{\,} \geq \frac{1}{\,} = a_{k+1}$

A.S.T (1) nonincreasing

$$a_k = \frac{1}{k} \geqslant \frac{1}{k+1} = a_{k+1}$$

(2) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$ ✓

→ Alternating Harmonic Series converges
(regular H.S. diverges)

Ex: (2) $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$

$\frac{3}{2} \overset{?}{\geq} \frac{4}{3}$

$9 \geq 8$

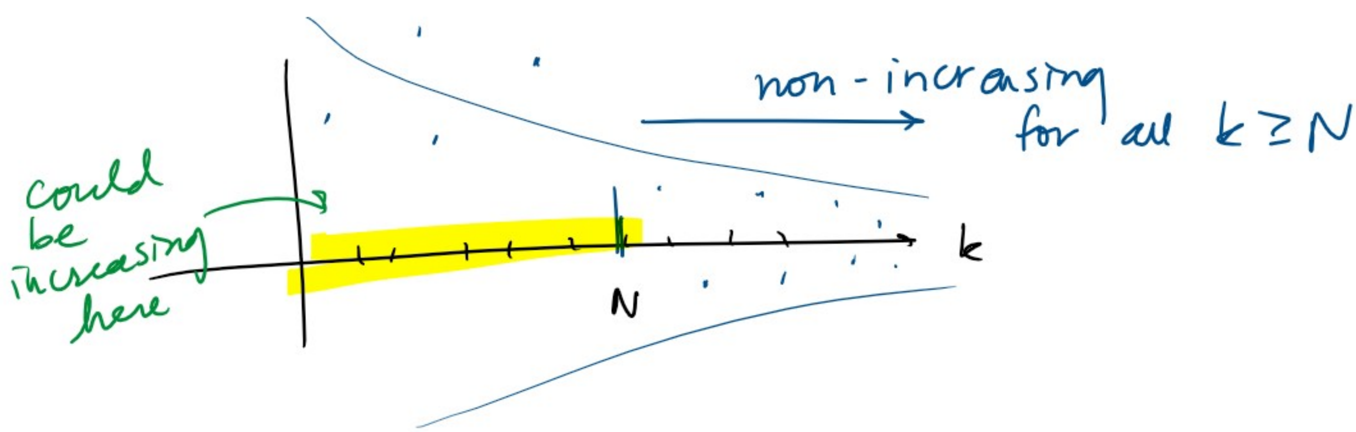
$$\sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{k+1}{k}\right)$$

Does this converge or diverge?

(1) nonincreasing? ✓ (for all k) $a_k = \frac{k+1}{k} \geqslant \frac{k+2}{k+1} = a_{k+1}$

(2) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+1}{k} = 1$ ✗

→ diverges



II. Estimating Alternate Series:

$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$

Suppose $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$

and (1) a_k are decreasing
(2) $\lim_{k \to \infty} a_k = 0$

$\Rightarrow$ A.S.T series converges to S

<u>Want</u>: to estimate S

partial sums

Remainder $|R_n| = |S - S_n|$

$$\boxed{\begin{array}{l}\underline{\text{Estimation Theorem:}} \\ |S - S_n| = |R_n| \le a_{n+1}\end{array}}$$

<u>Ex (1)</u>: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ Alt. Harm. Series $\longrightarrow$ converges

How many terms are needed to approximate S with error less than 10^{-6}?

Est. Thm: $|R_n| \le a_{n+1} = \frac{1}{n+1} < 10^{-6}$ set

$$n+1 > 10^6$$

need $n = 10^6$ terms

<u>Ex (2)</u>: Find S_3 and use R_3 to estimate S

$S = \sum^{\infty} \frac{(-1)^{k+1}}{\ldots}$

$$S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3+1}$$

$$S_3 = \frac{(-1)^{1+1}}{1^3+1} + \frac{(-1)^{2+1}}{2^3+1} + \frac{(-1)^{3+1}}{3^3+1}$$

$$= \frac{1}{2} - \frac{1}{9} + \frac{1}{28} = \frac{107}{252} \approx 0.4246$$

$$|R_3| \leq a_4 = \frac{1}{4^3+1} = \frac{1}{65} \approx 0.0153$$

$$S_3 - |R_3| \leq S \leq S_3 + |R_3|$$

$$0.4093 \leq S \leq 0.4399$$

III. Types of convergence:

Let $\sum a_k$ be any series

(could be
- positive terms
- alternating
- anything else)

<u>Def</u>: If $\sum |a_k|$ converges, then $\sum a_k$ <u>converges absolutely</u>

If $\sum |a_k|$ diverges (and $\sum a_k$ converges)
$\Rightarrow$ <u>conditional</u> <u>convergence</u>

Ex: (1) Alternating Harm. Series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \to \text{converges}$$

$$\sum_{k=1}^{\infty} \frac{1}{k} \to \text{diverges}$$

$\Longrightarrow$ A.H.S. converges conditionally

Ex (2): $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$

comparison test
$|\sin(k)| \le 1$

$$\sum_{k=1}^{\infty} \left|\frac{\sin(k)}{k^2}\right| \le \sum_{k=1}^{\infty} \frac{1}{k^2} \to \text{p-series } p = 2 > 1 \to \text{converges}$$

$\sum |a_k|$ converges

$$\longrightarrow \sum_{k=1}^{\infty} \frac{\sin(k)}{k^2} \text{ converges absolutely}$$

NOTE: What about other cases:

$\sum \lvert a_k \rvert$	$\sum a_k$	
converges	converges	$\Rightarrow$ absolute convergence
diverges	converges	$\Rightarrow$ conditional convergence
converges	diverges	X can't happen by comp. test

converges ...

happen by Comp. Test

diverge diverge $\Rightarrow$ divergence (can happen)

Ex: $\displaystyle\sum_{k=3}^{\infty} \frac{(-1)^k}{\ln(k)}$

(a) Does the series converge?

(i) non increasing $a_k = \frac{1}{\ln(k)} \geq \frac{1}{\ln(k+1)} = a_{k+1}$

(ii) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{\ln(k)} = 0$

Alternating Series converges

(b) Does it converge absolutely?

$$\sum_{k=3}^{\infty} \left| \frac{(-1)^k}{\ln(k)} \right| = \sum_{k=3}^{\infty} \frac{1}{\ln|k|}$$

Limit Comparison Test

$a_k = \frac{1}{\ln|k|}$ $\quad b_k = \frac{1}{k} \rightarrow$ diverges

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{\ln(k)} \cdot \left(\frac{k}{1}\right) = \lim_{k\to\infty} \frac{k}{\ln(k)} \ \frac{\infty}{\infty}$$

L'Hôpital's $= \lim_{k\to\infty} \frac{1}{\left(\frac{1}{k}\right)} = \lim_{k\to\infty} k = +\infty$

$L = \infty$, $\sum b_k$ diverges $\Rightarrow \sum \left|\frac{1}{\ln(k)}\right|$ ...

$L = \infty$, $\sum b_k$ diverges $\Rightarrow$ $\sum \left|\frac{1}{\ln(k)}\right|$ also diverges

$\rightarrow$ converges conditionally.