10.6: Alternating Series Test

☆ Warm Up: Use the Comparison Test to determine if the following series converges:

$$\sum_{k=1}^{\infty} \left(\frac{4k}{2k-3}\right)^k$$

(a) converges

$\boxed{\text{(b) diverges}}$

$$a_k = \left(\frac{2}{1-\frac{3}{2k}}\right)^k \qquad b_k = 2^k$$

geometric series
$r = 2 > 1$
diverges

$$a_k = \left(\frac{2}{1-\frac{3}{2k}}\right) \geq 2^k = b_k \longrightarrow \sum b_k \text{ diverges}$$

$\sum a_k$ also diverges

I. Alternating Series:

So far $\sum_{k=1}^{\infty} a_k \rightarrow a_k > 0$

An alternating series is an infinite series where the terms alternate positive and negative

first term pos

Ex: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Ex: $\sum_{k=1} \frac{}{k}$

→ Alternating Harmonic Series

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^2+1} \xrightarrow{\text{first term neg}} = -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \ldots$

Two Types:

math is same for both types

$$\begin{cases} \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} a_k, \quad a_k > 0 \\ \quad \Rightarrow \text{1st term positive} \\ \displaystyle\sum_{k=1}^{\infty} (-1)^{k} a_k, \quad a_k > 0 \\ \quad \Rightarrow \text{1st term negative} \end{cases}$$

Q: When does an alternating series converge?

<u>Alternating Series Test</u>:

$\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, where $a_k > 0$

⇒ converges provided

(1) terms are <u>non-increasing</u> in magnitude

$a_k \geq a_{k+1}$ for all $k > N$

(where N is a finite number)

(2) $\displaystyle\lim_{k \to \infty} a_k = 0$

Ex (1): Alt. Harm. Series $\displaystyle\sum^{\infty} (-1)^{k+1} \left(\frac{1}{k}\right)$

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$a_k = \frac{1}{k}$

A.S.T: (1) non-increasing ✓ for all k

$$\frac{1}{k} = a_k \geq a_{k+1} = \frac{1}{k+1}$$ ✓

(2) $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{k} = 0$ ✓

$\Rightarrow$ Alt. Harmonic Series converges

(Harmonic series diverges)

Ex (2): $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots$

$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k+1}{k}\right)$ Does this converge?

(1) non increasing? ✓ $\frac{k+1}{k}$ $a_k \overset{?}{\geq} a_{k+1} = \frac{k+2}{k+1}$

$$(k+1)^2 \overset{?}{\geq} k(k+2)$$

$$k^2+2k+1 \overset{?}{\geq} k^2+2k$$ ✓

(2) $\lim_{k \to \infty} a_k = ?$

$\lim_{k \to \infty} \frac{k+1}{k} = 1 \neq 0 \rightarrow$ diverges

II. Estimating Alternating Series

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Suppose $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$

and (1) a_k is decreasing

(2) $\lim_{k\to\infty} a_k = 0$

$\Rightarrow$ A.S.T. series converges to S

Want: to estimate S

S_n is partial sum

Remainder $|R_n| = |S - S_n|$

Estimation Theorem: (for alt. series that converges

$$|S - S_n| = |R_n| \leq a_{n+1}$$

Ex: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ Alt. Harm. Series $\Rightarrow$ converges

How many terms are need to approximate S with error less than 10^{-6}?

$$|R_n| \leq a_{n+1} = \frac{1}{n+1} \overset{\text{set}}{<} 10^{-6}$$

$$n+1 > 10^6$$

need $n = 10^6$ terms

Ex(2): Find S_3 and use R_3 to estimate S

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$$S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3+1}$$

$$S_3 = \frac{(-1)^{1+1}}{1^3+1} + \frac{(-1)^{2+1}}{2^3+1} + \frac{(-1)^{3+1}}{3^3+1}$$

$$= \frac{1}{2} - \frac{1}{9} + \frac{1}{28} = \frac{107}{252} \approx 0.4246$$

$$|R_3| \le a_4 = \frac{1}{4^3+1} = \frac{1}{65} \approx 0.0153$$

$$S_3 - |R_3| \le S \le S_3 + |R_3|$$

$$0.4093 \le S \le 0.4399$$

III. Types of Convergence:

Let $\sum a_k$ be any series

can be
- positive terms
- alternating series
- even more general

Def:

(1) If $\sum |a_k|$ converges, then $\sum a_k$ converges absolutely

converges absolutely

(2) If $\sum |a_k|$ diverges but $\sum a_k$ converges

$\Rightarrow$ converges conditionally

Ex (1): Alt. Harm. Series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \longrightarrow \text{A.S.T. converges}$$

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \longrightarrow \text{diverges (by Integral Test)}$$

$\longrightarrow$ converges conditionally

Ex (2): $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ $|\sin(k)| \le 1$

$$\sum_{k=1}^{\infty} \left| \frac{\sin(k)}{k^2} \right| \le \sum_{k=1}^{\infty} \frac{1}{k^2}$$

p-series
$p=2$
converges

$\uparrow$ converges (by the Comparison Test)

$\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ converges absolutely

NOTE: What about other cases?

$\sum \lvert a_k\rvert$	$\sum a_k$	
converges	converges	$\Rightarrow$ absolute convergence
diverges	converges	$\Rightarrow$ conditional convergence
~~converges~~	~~diverges~~	can't happen $a_k \le \lvert a_k\rvert$ $\sum \lvert a_k\rvert$ converges $\Rightarrow \sum a_k$ also converges!
diverges	diverges	can happen (diverges) no extra terminology

Ex (3): $\displaystyle\sum_{k=3}^{\infty} \frac{(-1)^k}{\ln(k)}$

(a) Does the alternating series converge?

A.S.T. (1) non-increasing ✓ $\quad \dfrac{1}{\ln(k)} = a_k \ge a_{k+1} = \dfrac{1}{\ln(k+1)}$ ✓

(2) $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{\ln(k)} = 0$ ✓

So, yes it converges by A.S.T.

(b) Does it converge absolutely?

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$$\sum_{k=3}^{\infty} \left| \frac{(-1)^{k+1}}{\ln|k|} \right| = \sum_{k=3}^{\infty} \frac{1}{\ln|k|}$$

Limit Comparison Test:

$$a_k = \frac{1}{\ln(k)} \qquad b_k = \frac{1}{k}$$

Harm. p-series $p=1$ diverges

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{\ln(k)} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{k}{\ln(k)} \quad \frac{\infty}{\infty}$$

L'Hopital's $= \lim_{k\to\infty} \frac{1}{\left(\frac{1}{k}\right)} = \lim_{k\to\infty} k = \infty$

$L = \infty$ $\quad \sum b_k$ diverges $\Rightarrow$ $\sum a_k$ diverges

$\longrightarrow$ converges conditionally

$$\sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{\ln(k)}$$