Lesson 25 - MA 166 LEC 200

10.7: Ratio Test & Root Test

Announcements:
Final Exam Mon May 2nd
@ 10:30 am – 12:30 pm in ELLT

Warm Up: Determine the convergence of the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{3^k + 4^k}$$

(a) converges absolutely

(b) converges conditionally

(c) diverges

if $\sum |a_k|$

$$\left| \frac{\sin(k)}{3^k+4^k} \right| \le \left| \frac{1}{3^k+4^k} \right| < \frac{1}{3^k} = b_k$$

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{3^k}$$ Geometric $r = \frac{1}{3} < 1$ Converges

Comparison Test

$|a_k| < b_k$ $\quad \sum b_k$ converges $\Rightarrow \sum |a_k|$ converges

$\Rightarrow \sum a_k$ converges absolutely.

So far, the Converges Tests

Test	Conditions/Limitations
	only shows divergence

Test	
Divergence Test	only shows divergence
Integral Test	$f(x)$ st. $f(k) = a_k > 0$ and $f(x)$ is continuous, positive, decreasing
Comparison Test	Need $a_k > 0$
Limit Comparison Test	Need $a_k > 0$
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, $a_k > 0$; $\sum_{k=1}^{\infty} (-1)^k a_k$, $a_k > 0$
Absolute Convergence Test	any $\sum_{k=1}^{\infty} a_k$

I. Ratio Test:

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series where $a_k \neq 0$ for all k

$$\text{Let} \quad r = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

Then:

r		convergence
$r < 1$	$\Rightarrow$	$\sum a_k$ converges absolutely

$r = 1 \Rightarrow$ Inconclusive

$r > 1$ $(= \infty)$ $\Rightarrow$ $\sum a_k$ diverges

DNE $\Rightarrow$ Inconclusive

Ex(1): $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k!}$ $\qquad a_k = \dfrac{10^k}{k!}$

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{\left| \frac{10^{k+1}}{(k+1)!} \right|}{\frac{10^k}{k!}}$$

$$= \lim_{k\to\infty} \frac{10^{\cancel{k}+1}}{(k+1)\cancel{!}} \cdot \frac{\cancel{k!}}{\cancel{10^k}} = \lim_{k\to\infty} \frac{10}{k+1} = 0$$

$$\frac{k! = \cancel{k} \cdot \cancel{(k-1)} \cancel{(k-2)} \cdots \cancel{2} \cdot \cancel{1}}{(k+1)! = (k+1) \cancel{(k)} \cancel{(k-1)} \cdots \cancel{2} \cdot \cancel{1}} = \frac{1}{k+1}$$

$r = 0 \Rightarrow \sum a_k$ converges absolutely by the Ratio Test.

Ex(2): $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k\, k^k}{k!}$

$$\lim \left| \frac{\cancel{(-1)^{k+1}} (k+1)^{(k+1)}}{\quad} \right|$$

Ratio Test : $r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{\left|\frac{\cancel{(-1)^{k+1}}\,(k+1)^{k+1}}{(k+1)!}\right|}{\left|\frac{\cancel{(-1)^k}\,k^k}{k!}\right|}$

$$= \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)\cancel{!}} \cdot \frac{\cancel{k!}}{k^k} = \lim_{k\to\infty} \frac{(k+1)^{k+\cancel{1}}}{\cancel{(k+1)}} \cdot \frac{1}{k^k}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k$$

$$= \lim_{k\to\infty} e^{\ln\left[\left(1+\frac{1}{k}\right)^k\right]}$$

$$= e^{\boxed{\lim_{k\to\infty} k \ln\left[1+\frac{1}{k}\right]}} \rightarrow 1$$

$$\lim_{k\to\infty} k \ln\left[1+\frac{1}{k}\right] \qquad \infty \cdot 0$$

$$= \lim_{k\to\infty} \frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}} \qquad \frac{0}{0} \quad \text{L'Hopital's}$$

$$= \lim_{k\to\infty} \frac{\frac{1}{\left(1+\frac{1}{k}\right)} \cdot \cancel{-\frac{1}{k^2}}}{\cancel{-\frac{1}{k^2}}} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1$$

$$= e^{\boxed{}} = e^1$$

$r = e > 1 \implies \sum a_k$ diverges by the Ratio Test

$r = e > 1 \Longrightarrow \sum$ ...
by the Ratio Test

II. Root Test:

Let $\sum a_k$ be an infinite series

Let $\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|}$

ρ	Convergence
$\rho < 1$	$\sum a_k$ converges absolutely
$\rho = 1$	Inconclusive
$\rho > 1$ $(=\infty)$	$\sum a_k$ diverges
DNE	Inconclusive

Ex(3): $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3-4k^2}{7k^2-6}\right)^k$ ⟵ suggests we use Root Test

$$\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left|\left(\frac{3-4k^2}{7k^2-6}\right)^k\right|}$$

$$= \lim_{k\to\infty} \frac{4k^2-3}{7k^2-6}$$ ⟵ positive for k large

$$= \lim_{k\to\infty} \frac{4 - 3\left(\frac{1}{k^2}\right)}{7 - 6\left(\frac{1}{k^2}\right)} = \frac{4}{7}$$

$$\rho = \frac{4}{7} < 1 \qquad \sum a_k \text{ converges absolutely}$$

Ex (4): $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^k}{k^{10}}$ Does this converge or diverge?

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left|\frac{(-2)^k}{k^{10}}\right|} = \lim_{k\to\infty} \frac{2}{k^{10/k}}$$

$$= 2 \lim_{k\to\infty} k^{-10/k} \qquad \infty^0$$

$$= 2 \lim_{k\to\infty} e^{\ln\left[k^{-10/k}\right]} = 2e^{\boxed{\lim_{k\to\infty} -\frac{10}{k}\ln(k)}}$$

$$-10 \lim_{k\to\infty} \frac{\ln(k)}{k} \qquad \frac{\infty}{\infty} \qquad \text{L'Hopital's}$$

$$= -10 \lim_{k\to\infty} \frac{\frac{1}{k}}{1} = 0$$

$$\rho = 2e^{\boxed{}} = 2e^0 = 2 > 1$$

$\Rightarrow \sum a_k$ diverges by the Root Test

$\underline{\text{Ex (5)}}$: $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k} (k^2+4)$

$\underline{\text{Q}}$: Ratio Test or Root Test

$\boxed{\text{Root Test}}$

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left| \cancel{(-1)^{k+1}}^{\,1} e^{-k} (k^2+4) \right|}$$

$$= \lim_{k\to\infty} e^{-1} (k^2+4)^{1/k}$$

$$= \frac{1}{e} \lim_{k\to\infty} (k^2+4)^{1/k} \qquad \infty^0$$

$$= \frac{1}{e} e^{\lim_{k\to\infty} \ln\left[(k^2+4)^{1/k}\right]}$$

$$= \frac{1}{e} e^{\lim_{k\to\infty} \frac{1}{k} \ln(k^2+4)} \leftarrow \text{L'Hopital's}$$

$$= \frac{1}{e} e^{\lim_{k\to\infty} \frac{\left(\frac{1}{k^2+4}\right)\cdot 2k}{1}} \to 0 \qquad = \frac{1}{e}\cdot e^0$$

$\rho = \frac{1}{e} < 1 \implies \sum a_k$ converges absolutely