Lesson 25 - MA 166 LEC 286

10.7: Ratio Test & Root Test

Announcements:
Final Exam Mon May 2nd
@ 10:30 am – 12:30 pm in ELLT

Warm Up: Determine the convergence of the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{3^k + 4^k}$$

$\boxed{\text{(a) converges absolutely}}$ (b) converges conditionally (c) diverges

$\downarrow$

$\sum |a_k|$ converge

$$\left| \frac{\sin(k)}{3^k+4^k} \right| \le \left| \frac{1}{3^k+4^k} \right| < \frac{1}{3^k} = b_k$$

$$\sum b_k = \sum_{k=1}^{\infty} \frac{1}{3^k}$$

Geometric series $r = \frac{1}{3} < 1$ converges

$|a_k| < b_k$

$\sum b_k$ converges $\xrightarrow{\text{Comparison Test}}$ $\sum |a_k|$ converges

$\rightarrow \sum a_k$ converges absolutely

Tests	Conditions / Limitations
Divergence Test	only shows divergence

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Integral Test	$f(x)$ so that $f(k) = a_k > 0$ $f(x)$ is continuous, positive, decreasing
Comparison Test	$\sum a_k$ $\quad a_k > 0$
Limit Comparison Test	$a_k > 0$
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, $a_k > 0$, $\sum_{k=1}^{\infty} (-1)^k a_k$, $a_k > 0$
Absolute Convergence Test	works for any $\sum_{k=1}^{\infty} a_k$

I. Ratio Test:

Let $\sum a_k$ be an infinite series where $a_k \neq 0$ for all k

Let $r = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$

r		convergence
$r < 1$	$\Rightarrow$	$\sum a_k$ converges absolutely
$r = 1$	$\Rightarrow$	Inconclusive

$r=1$	$\Rightarrow$	Inconclusive
$r>1$ $(=\infty)$	$\Rightarrow$	$\sum a_k$ diverges
DNE	$\Rightarrow$	Inconclusive

Ex(1): $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k!}$ $\qquad a_k = \dfrac{10^k}{k!}$

Ratio Test

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{\left|\frac{10^{k+1}}{(k+1)!}\right|}{\left|\frac{10^k}{k!}\right|}$$

$$= \lim_{k\to\infty} \frac{10^{\cancel{k}+1}}{(k+1)\cancel{!}} \cdot \frac{\cancel{k!}}{\cancel{10^k}}$$

$$\frac{k!}{(k+1)!} = \frac{\cancel{k}(\cancel{k-1})(\cancel{k-2})\cdots\cancel{2}\cdot\cancel{1}}{(k+1)(\cancel{k})(\cancel{k-1})\cdots\cancel{2}\cdot\cancel{1}} = \frac{1}{k+1}$$

$$= \lim_{k\to\infty} \frac{10}{k+1} = 0$$

$r = 0 < 1$

$\Rightarrow$ Ratio Test

$\sum a_k$ converges absolutely.

Ex(2): $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^k}{k!}$

$\left|(-1)^{k+1}(k+1)^{k+1}\right|$

$k=1$ $\qquad k!$

Ratio Test

$$r = \lim_{k\to\infty} \frac{\left|\dfrac{\cancel{(-1)^{k+1}}^{\,1}(k+1)^{k+1}}{(k+1)!}\right|}{\left|\dfrac{\cancel{(-1)^{k}}^{\,1}\,k^{k}}{k!}\right|}$$

$$= \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)\cancel{!}} \cdot \frac{\cancel{k!}}{k^k} = \lim_{k\to\infty} \frac{(k+1)^{k+\cancel{1}}}{\cancel{(k+1)}} \cdot \frac{1}{k^k}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k$$

$x = \left(1+\frac{1}{k}\right)^k$

$x = e^{\ln(x)}$

$$= \lim_{k\to\infty} e^{\ln\left[\left(1+\frac{1}{k}\right)^k\right]}$$

$$= e^{\boxed{\lim_{k\to\infty} k\ln\left(1+\frac{1}{k}\right)}} \qquad \infty \cdot 0$$

$$\lim_{k\to\infty} k\ln\left(1+\frac{1}{k}\right) = \lim_{k\to\infty} \frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}} \qquad \frac{0}{0}$$

L'Hopital's $$= \lim_{k\to\infty} \frac{\frac{1}{\left(1+\frac{1}{k}\right)} \cdot \cancel{\frac{-1}{k^2}}}{\cancel{\frac{-1}{k^2}}}$$

$$= \lim_{k\to\infty} \left(\frac{1}{1+\frac{1}{k}}\right) = 1$$

$$r = e^{\boxed{}} = e^{1} = e > 1$$

Ratio Test

$\sum a_k$ <u>diverges</u>

II. Root Test:

Let $\sum a_k$ be an infinite series

$$\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|}$$

ρ		convergence
$\rho < 1$	$\Rightarrow$	$\sum a_k$ converges absolutely
$\rho = 1$	$\Rightarrow$	Inconclusive
$\rho > 1$ $(=\infty)$	$\Rightarrow$	$\sum a_k$ diverges
DNE	$\Rightarrow$	Inconclusive

Ex (3): $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3-4k^2}{7k^2-6}\right)^{k}$ ← suggests we use the Root Test

Root Test: $\displaystyle \rho = \lim_{k\to\infty} \sqrt[k]{\left|\left(\frac{3-4k^2}{7k^2-6}\right)^k\right|}$

$$= \lim_{k\to\infty} \sqrt[\cancel{k}]{\left|\frac{3-4k^2}{7k^2-6}\right|^{\cancel{k}}}$$

neg if $k>1$

$$= \lim \frac{|3-4k^2|}{\quad} = \lim \frac{4k^2-3}{\quad}$$

$$= \lim_{k\to\infty} \frac{|3-4k^2|}{|7k^2-6|} = \lim_{k\to\infty} \frac{4k^2-3}{7k^2-6}$$

$$= \lim_{k\to\infty} \frac{4-\frac{3}{k^2}}{7-\frac{6}{k^2}} = \frac{4}{7}$$

$\rho = 4/7 < 1$

By the root test $\sum a_k$ converges absolutely

Ex (4): $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^k}{k^{10}}$ Does this converge or diverge?

Root Test

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left|\frac{(-2)^k}{k^{10}}\right|} = \lim_{k\to\infty} \frac{2}{k^{10/k}}$$

$$= 2\lim_{k\to\infty} k^{-10/k} = 2\lim_{k\to\infty} e^{\ln\left[k^{-10/k}\right]}$$

$$= 2e^{\boxed{\lim_{k\to\infty} \frac{-10}{k}\ln(k)}}$$

$$-10\lim_{k\to\infty} \frac{\ln(k)}{k} \quad \frac{\infty}{\infty} \to \text{L'Hopital's}$$

$$= -10\lim_{k\to\infty} \frac{\frac{1}{k}}{1} = 0$$

$$\rho = 2e^{\Box} = 2e^{0} = 2 > 1$$

Root Test $\Rightarrow$ $\sum a_k$ diverges

$\sum^{\infty} (-1)^{k+1}$ $-k$ $(1, 2, 4)$ $-k \leftarrow$ $k \mapsto k+1$

Ex (5): $\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k} (k^2+4)$

$e^{-k} \leftarrow \quad k \mapsto k+1$

$e^{-(k+1)}$

Q: Ratio Test or Root Test

Root Test

$$\rho = \lim_{k\to\infty} \sqrt[k]{\left| \cancel{(-1)^{k+1}} e^{-k} (k^2+4) \right|}$$

$$= \lim_{k\to\infty} e^{-1} (k^2+4)^{1/k}$$

$$= \frac{1}{e} \lim_{k\to\infty} (k^2+4)^{1/k}$$

Ratio Test

$$r = \lim_{k\to\infty} \frac{\left| \cancel{(-1)^{k+2}} e^{-(\cancel{k}+1)} \left[(k+1)^2+4 \right] \right|}{\left| \cancel{(-1)^{k+1}} \cancel{e^{-k}} \left[k^2+4 \right] \right|}$$

$$= \lim_{k\to\infty} e^{-1} \left[\frac{(k+1)^2+4}{k^2+4} \right]$$

$$= \frac{1}{e} \lim_{k\to\infty} \frac{k^2+2k+5}{k^2+4} \to 1$$

$r = \frac{1}{e} < 1 \Rightarrow$ Ratio Test

$\sum a_k$ converges absolutely

Both Test work and give same result

and give same

$$\rho = \frac{1}{e}$$