Lesson 26 - MA 166 LEC 200

10.8: Choosing a Convergence Test

Announcements:

Exam 2 Booklets returned Thursday in Recitation

Answer Keys will be posted

★ **Warm Up:** Use the Ratio Test or Root Test to determine if the series converges:

$$\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$$

(a) converges absolutely (b) Diverges (c) Inconclusive

Ratio Test: $r = \lim_{k\to\infty} \dfrac{\left|\dfrac{2^{k+1}(k+1)!}{(k+1)^{k+1}}\right|}{\left|\dfrac{2^k k!}{k^k}\right|} = \lim_{k\to\infty} \dfrac{2^{\cancel{k+1}}(k+1)\cancel{!}\, k^k}{\cancel{2^k}\, \cancel{k!}\, (k+1)^{k+1}}$

$$= 2\lim_{k\to\infty} \frac{\cancel{(k+1)}k^k}{(k+1)^{\cancel{k+1}}} = 2\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k$$

$$= 2\lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{k}}\right)^k = 2e^{\boxed{\lim_{k\to\infty} \ln\left[\left(\frac{1}{1+\frac{1}{k}}\right)^k\right]}}$$

$$\lim_{k\to\infty} k\ln\left(\frac{1}{1+\frac{1}{k}}\right) = \lim_{k\to\infty} \frac{-\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}}$$

L'Hopitals

$$= \lim_{k\to\infty} \frac{-\frac{1}{1+\frac{1}{k}}\cancel{\left(-\frac{1}{k^2}\right)}}{\cancel{-\frac{1}{k^2}}} =$$

$$= \lim_{k\to\infty} -\frac{1}{1+\frac{1}{k}} = -1$$

$$r = 2e^{\square} = 2e^{-1} = \frac{2}{e} < 1$$

converges absolutely

$= \lim_{k \to \infty} \frac{\quad}{1+\frac{1}{k}} = 1$

$r = 2e^{\square} = 2e^{-1} = \frac{2}{e} < 1$ $\boxed{\text{converges absolutely}}$

Aside: If $\frac{3^k k!}{k^k} \rightarrow r = \frac{3}{e} > 1$

I. Review of Convergence Tests:

Series/Test	Form	Convergence	Divergence	Comments
Geometric Series	$\sum_{k=1}^{\infty} ar^k$	$\lvert r \rvert < 1$	$\lvert r \rvert \geq 1$	if $\lvert r \rvert < 1$ $\rightarrow \frac{a}{1-r}$
Divergence Test	$\sum_{k=1}^{\infty} a_k$	X	$\lim_{k\to\infty} a_k \neq 0$	Can't be used to show conv.
Integral Test	$\sum_{k=1}^{\infty} a_k$, $a_k = f(k)$ f is cts, pos, and decreasing	$\int_1^{\infty} f(x)dx$ converges	$\int_1^{\infty} f(x)dx$ diverges	value of the integral $\neq$ value of series
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$ $(a_k \neq 0)$	$r = \lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert < 1$	$r > 1$	Inconclusive if $r = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\rho = \lim_{k\to\infty} \sqrt[k]{\lvert a_k \rvert} < 1$	$\rho > 1$	Inconclusive if $\rho = 1$
Comparison Test	$\sum^{\infty} a_k$, $a_k > 0$	$a_k \leq b_k$ and $\sum b_k$ converges	$a_k \geq b_k$ and $\sum b_k$ diverges	Need to find b_k

Comparison Test	$\sum_{k=1}^{\infty} a_k,\ a_k>0$	$a_k \le b_k$ and $\sum b_k$ converges	$a_k \ge b_k$ and $\sum b_k$ diverges	Need to find b_k
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k,\ a_k>0$ $L=\lim_{k\to\infty}\left\lvert\frac{a_k}{b_k}\right\rvert$	$0 \le L < \infty$ $\sum b_k$ converges	$L>0$ $\sum b_k$ diverges	
Alternating Series Test	$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ $a_k>0$	$a_k \geq a_{k+1}$ $\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \neq 0$	Remainder $\lvert R_n\rvert \le \lvert a_{n+1}\rvert$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$	$\sum_{k=1}^{\infty}\lvert a_k\rvert$ converges		

p 701 in Textbook

II. General Strategy:

1. What does the given series look like?
   Find the closest model series b_k
   → converge?
   → diverge?

2. Justify using a test

Ex: (1) $$\sum_{k=1}^{\infty} \frac{2^k + \boxed{\cos(\pi k)\sqrt{k}}}{3^{k+1}}$$

Looks like $\sum \frac{2^k}{3^{k+1}}$ → Geometric series $r=\frac{2}{3}$ converges

How to ... $\left\lvert \frac{\cos(\pi k)\sqrt{k}}{3^{k+1}} \right\rvert$ $\quad \lvert\cos(\pi k)\rvert \le 1$

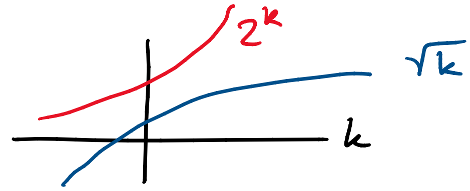
$\sqrt{k} \le 2^k$

How to deal $\boxed{}$

$$\left|\frac{\cos(\pi k)\sqrt{k}}{3^{k+1}}\right| \le \frac{2^k}{3^{k+1}}$$

$|\cos(\pi k)| \le 1$

$\sqrt{k} \le 2^k$



$$|a_k| = \left|\frac{2^k + \cos(\pi k)\sqrt{k}}{3^{k+1}}\right| \le \left|\frac{2^k}{3^{k+1}}\right| + \left|\frac{\cos(\pi k)\sqrt{k}}{3^{k+1}}\right|$$

$$\le \frac{2^k}{3^{k+1}} + \frac{2^k}{3^{k+1}} = 2\left(\frac{2^k}{3^{k+1}}\right) = b_k$$

$|a_k| \le |b_k| \longrightarrow$ Comparison Test

$\hookrightarrow$ converges b/c Geom series $r<1$

By Comp. Test $\sum |a_k|$ also converges.

So $\sum a_k$ conv. absolutely

Ex (2): $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^2-6k+9}}$

Q: what does this series look like

$$a_k = \frac{1}{\sqrt[4]{k^2-6k+9}} \sim \frac{1}{\sqrt[4]{k^2}} = \frac{1}{k^{1/2}} = b_k$$

when k is large

p-series $p=\frac{1}{2}$ diverges

expect $\sum a_k$ to diverge

Q: How to Justify? Limit Comparison Test

$$L = \lim_{k\to\infty} \left| \frac{\frac{1}{\sqrt[4]{k^2-6k+9}}}{\frac{1}{k^{1/2}}} \right| = \lim_{k\to\infty} \sqrt[4]{\frac{k^2}{k^2-6k+9}} = 1$$

$L=1$ and $\sum b_k$ diverges $\Rightarrow$ $\sum a_k$ diverges

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Ex (3): $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-2k}$

$a_k = k^2 e^{-2k}$ ← decreasing

↑ increasing

Q: Is a_k decreasing?

$$\lim_{k\to\infty} \frac{k^2}{e^{2k}} \overset{L'H}{=} \lim_{k\to\infty} \frac{2k}{2e^{2k}} \overset{L'H}{=} \lim_{k\to\infty} \frac{2}{4e^{2k}} = 0$$

Divergence Test Inconclusive

Q: What does the series look like

$a_k = \frac{k^2}{e^{2k}}$ $\quad b_k = \frac{1}{e^{2k}}$ Geometric $r = \frac{1}{e^2} \Rightarrow$ converges

but $a_k \geq b_k$

No Obvious Comparison Test

→ Ratio or Root Test

$a_k = \frac{k^2}{e^{2k}}$ ← no k in exponent

$$r = \lim_{k\to\infty} \frac{(k+1)^2}{e^{2\cancel{(k+1)}}} \cdot \frac{e^{\cancel{2k}}}{k^2} = \frac{1}{e^2} \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 = \frac{1}{e^2}$$

$r = \frac{1}{e^2} < 1$, $\sum a_k$ converges

<u>Ex (4)</u>: $\sum_{k=1}^{\infty} \left(1 - \frac{10}{k}\right)^k$

Root Test : $\rho = \lim_{k\to\infty} \sqrt[k]{\left|\left(1-\frac{10}{k}\right)^k\right|} = \lim_{k\to\infty} 1 - \frac{10}{k} = 1$

Inconclusive b/c $\rho = 1$

$\left(1 - \frac{10}{k}\right) \underset{\substack{\text{when} \\ k \text{ is} \\ \text{large}}}{\sim} 1$

$\left(1 - \frac{10}{k}\right)^k \overset{?}{\sim} 1^k$

$\left(1 - \frac{10}{k}\right) < 1$

so $\left(1 - \frac{10}{k}\right)^k < 1$

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(1 - \frac{10}{k}\right)^k = e^{\boxed{\lim_{k\to\infty} \ln\left[\left(1-\frac{10}{k}\right)^k\right]}}$

$\lim_{k\to\infty} k \ln\left(1 - \frac{10}{k}\right) = \lim_{k\to\infty} \frac{\ln\left(1-\frac{10}{k}\right)}{\frac{1}{k}}$

$L'H = \lim_{k\to\infty} \frac{\frac{1}{1-\frac{10}{k}} \cdot \frac{+10}{\cancel{k^2}}}{\frac{-1}{\cancel{k^2}}}$

$= \lim_{k\to\infty} \frac{-10}{1 - \frac{10}{k}} = -10$

$\lim_{k\to\infty} a_k = e^{\boxed{}} = e^{-10} \neq 0$ so By Div. Test $\sum a_k$ diverges.