Lesson 26 - MA 166 LEC 286

10.8: Choosing a Convergence Test

Announcements:

Exam 2 Booklets returned Thursday in Recitation

Answer Keys will be posted

★Warm Up: Use the Ratio Test or Root Test to determine if the series converges:

$$\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$$

$\boxed{\text{(a) Converges absolutely}}$ (b) Diverges (c) Inconclusive

Ratio Test

$$r = \lim_{k\to\infty} \frac{2^{\cancel{k+1}}(k+1)\cancel{!}}{(k+1)^{k+1}} \cdot \frac{k^k}{\cancel{2^k}\,\cancel{k!}}$$

$$= 2\lim_{k\to\infty} \frac{\cancel{(k+1)}}{(k+1)^{k\cancel{+1}}} \cdot k^k = 2\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k$$

$$= 2e^{\boxed{\lim_{k\to\infty} \ln\left[\left(\frac{k}{k+1}\right)^k\right]}}$$

$$\lim_{k\to\infty} k\ln\left(\frac{1}{1+\frac{1}{k}}\right) = \lim_{k\to\infty} -k\ln\left(1+\frac{1}{k}\right)$$

$$= \lim_{k\to\infty} \frac{-\ln\left(1+\frac{1}{k}\right)}{\left(\frac{1}{k}\right)} \overset{L'H}{=} \lim_{k\to\infty} \frac{-\left(\frac{1}{1+\frac{1}{k}}\right)\cancel{\left(-\frac{1}{k^2}\right)}}{\cancel{\left(-\frac{1}{k^2}\right)}}$$

$$= \lim_{k\to\infty} \frac{-1}{1+\frac{1}{k}} = -1$$

$$r = 2e^{\boxed{}} = 2e^{-1} = \frac{2}{e} < 1$$

By the Ratio Test <u>converges</u>.

I. Review $\quad$ p 701 in Textbook

Series / Test	Form	Convergence	Divergence	Comments
Geometric Series	$\sum_{k=1}^{\infty} ar^k$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	if $\lvert r\rvert < 1$ $\rightarrow \frac{a}{1-r}$
Divergence Test	$\sum_{k=1}^{\infty} a_k$	X	$\lim_{k\to\infty} a_k \neq 0$	
Integral Test	$\sum_{k=1}^{\infty} a_k$, $a_k > 0$ $f(k) = a_k$ $f(x)$ is cts, pos, + decreasing	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	the value of the integral $\neq$ the value of the series
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	
Ratio Test	$\sum_{k=1}^{\infty} a_k$ $a_k \neq 0$	$r = \lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert < 1$	$r > 1$	Inconclusive if $r = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\rho = \lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\rho > 1$	Inconclusive if $\rho = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$ $a_k > 0$	$a_k \leq b_k$ $\sum b_k$ converges	$a_k \geq b_k$ $\sum b_k$ diverges	
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, $a_k > 0$ $L = \lim_{k\to\infty} \left\lvert \frac{a_k}{b_k} \right\rvert$	$0 \leq L < \infty$ $\sum b_k$ converge	$L > 0$ $\sum b_k$ diverge	
[illegible]	$\sum^{\infty}$ ... $k+1$	$a_k \geq a_{k+1}$	$\lim a_k \neq 0$	Remainder

Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, $a_k > 0$	$a_k \geq a_{k+1}$, $\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \neq 0$	Remainder $\lvert R_n \rvert \leq \lvert a_{n+1} \rvert$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$	$\sum_{k=1}^{\infty} \lvert a_k \rvert$ converges	X	X

II. General Strategy:

1. What does the given series look like
   ↳ Find the closest model series b_k
   - converge?
   - diverge?

2. Justify using a test

Ex (1): $\displaystyle\sum_{k=2}^{\infty} \sqrt[3]{\frac{k^2-1}{k^8+4}}$

$$a_k = \sqrt[3]{\frac{k^2-1}{k^8+4}} \sim \sqrt[3]{\frac{k^2}{k^8}} = \sqrt[3]{\frac{1}{k^6}} = \frac{1}{k^2} = b_k$$

when k is large

p-series $p = 2 > 1$

converges

expect a_k to also converge

Q: What test to use? → Limit Comparison Test

$$L = \lim_{k\to\infty} \left| \frac{a_k}{b_k} \right| = \lim_{k\to\infty} \frac{\sqrt[3]{\frac{k^2-1}{k^8+4}}}{\left(\frac{1}{k^2}\right)}$$

$$L = \lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| = \lim_{k\to\infty} \frac{\quad k^8+4}{\left(\frac{1}{k^2}\right)}$$

$$= \lim_{k\to\infty} k^2 \sqrt[3]{\frac{k^2-1}{k^8+4}} = \lim_{k\to\infty} \sqrt[3]{\frac{k^8-k^6}{k^8+4}} = 1$$

$L=1$ $\quad \sum b_k$ converges $\Rightarrow$ $\sum a_k$ also converges

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Ex(2): $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-2k}$ $\qquad a_k = k^2 e^{-2k}$

k^2: increasing $\qquad e^{-2k}$: decreasing

Q: Is a_k decreasing? $\to$ Div Test

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{e^{2k}} \overset{L'H}{=} \lim_{k\to\infty} \frac{2k}{2e^{2k}} \quad \swarrow L'H$$

$$= \lim_{k\to\infty} \frac{2}{4e^{2k}} = 0 \rightarrow \text{Div Test is inconclusive}$$

Q: What does the series look like?

$a_k = \dfrac{k^2}{e^{2k}}$ $\qquad b_k = \dfrac{1}{e^{2k}}$ $\qquad$ Geometric Series $r = \left(\frac{1}{e^2}\right) < 1$ converges

$a_k \geq b_k$ $\quad \sum b_k$ converges

Comparison Test Inconclusive

No Obvious Comparison $\rightarrow$ $\boxed{\text{Ratio}}$ or Root

$a_k = \dfrac{k^2}{e^{2k}}$ $\leftarrow$ no power of k in numerator

$$\frac{(k+1)^2}{\quad} \cdot \frac{\cancel{e^{2k}}}{\quad} = \frac{1}{\quad} \lim \left(\frac{k+1}{\quad}\right)^2 = \frac{1}{\quad}$$

$$r = \lim_{k\to\infty} \frac{(k+1)^2}{e^{\cancel{2(k+1)}}} \cdot \frac{\cancel{e^{2k}}}{k^2} = \frac{1}{e^2} \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 = \frac{1}{e^2}$$

$$r = \frac{1}{e^2} < 1 \implies \sum a_k \text{ converges}$$

Ex(3): $\displaystyle\sum_{k=1}^{\infty} \left(1 - \frac{10}{k}\right)^{k}$ ← Root Test

Root Test $\displaystyle\rho = \lim_{k\to\infty} \sqrt[k]{\left|\left(1-\frac{10}{k}\right)^k\right|} = \lim_{k\to\infty} 1 - \frac{10}{k} = 1$

$\rho = 1 \rightarrow$ Inconclusive

Q: What does the series look like

$\left(1 - \frac{10}{k}\right) \underset{\text{when } k \text{ is large}}{\sim} 1$ $\qquad \left(1-\frac{10}{k}\right) < 1$

$\left(1 - \frac{10}{k}\right)^k \overset{?}{\sim} 1^k$ $\qquad \left(1-\frac{10}{k}\right)$ could be smaller than 1

Div Test

$$\lim_{k\to\infty} \left(1 - \frac{10}{k}\right)^k = e^{\boxed{\lim_{k\to\infty} \ln\left[\left(1-\frac{10}{k}\right)^k\right]}}$$

$$\lim_{k\to\infty} k \ln\left(1 - \frac{10}{k}\right) = \lim_{k\to\infty} \frac{\ln\left(1-\frac{10}{k}\right)}{\left(\frac{1}{k}\right)} \quad L'H$$

$$= \lim_{k\to\infty} \frac{\frac{1}{\left(1-\frac{10}{k}\right)} \cdot -10 \cancel{\left(\frac{-1}{k^2}\right)}}{\cancel{\left(\frac{-1}{k^2}\right)}} = \lim_{k\to\infty} \frac{-10}{1 - \frac{10}{k}} = -10$$

$= e^{\square} = e^{-10} \neq 0$

By the Div Test

$\sum a_k$ diverges