Lesson 27 - MA 166 LEC 200

Announcements:
Exam 2 Answer Key posted on Brightspace

11.1: Approximating Functions with Polynomials

★ **Warm Up:** Determine if the following series converges

$$\sum_{k=1}^{\infty} \frac{\cot^{-1}\left(\frac{1}{k}\right)}{k^2}$$

$\boxed{\text{(a) converges}}$ (b) diverges

Comparison Test

$b_k = \frac{1}{k^2}$ ← p-series, $p=2$, converges

I. Power Series:

So far $\sum_{k=1}^{\infty} a_k$ where a_k are real numbers

Q: What if ~~if~~ $a_k = a_k(x)$ is a function of x?

Def: A power series is an infinite series

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n + \dots$$

or more generally

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + \dots + c_n (x-a)^n + \dots$$

a – center of the series

... real numbers

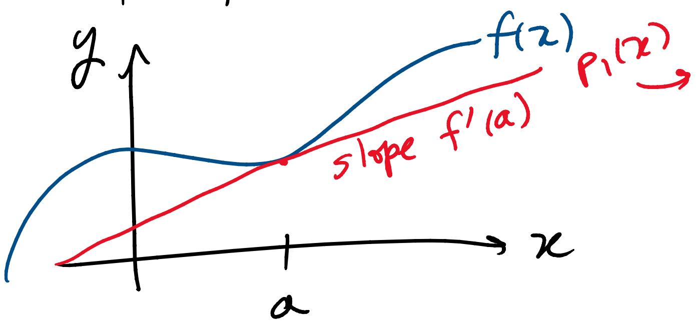
a – center of the series

c_k – coefficents are constant real numbers

GOAL: Use power series to approximate a function $f(x)$

II. Linear Approximations:

In Calc I, you found the tangent line to $f(x)$ at point $x=a$



$$p_1(x) = f(a) + f'(a)(x-a)$$

NOTE: We call $p_1(x)$ a linear approximation b/c it is a line

To make this approximation better, let's add another term:

$$p_2(x) = \underbrace{f(a) + f'(a)(x-a)}_{p_1(x)} + \underbrace{c_2(x-a)^2}_{\text{quadratic term}}$$

$p_2(x)$ – quadratic polynomial
→ Quadratic Approximation

Q: What is c_2?

Notice that:

$$p_2(a) = f(a) + f'(a)\cancelto{0}{(a-a)} + c_2\cancelto{0}{(a-a)}^2$$

Notice that

$$p_2(a) = f(a) + f'(a)\cancel{(a-a)} + c_2\cancel{(a-a)}$$

$$\boxed{p_2(a) = f(a)}$$

$$p_2'(a) = \left[0 + f'(a)(1) + c_2 \cdot 2(x-a)\right]\Big|_{x=a}$$

$$= f'(a) + 2c_2\underbrace{\cancel{(a-a)}}_{\to 0}$$

$$\boxed{p_2'(a) = f'(a)}$$

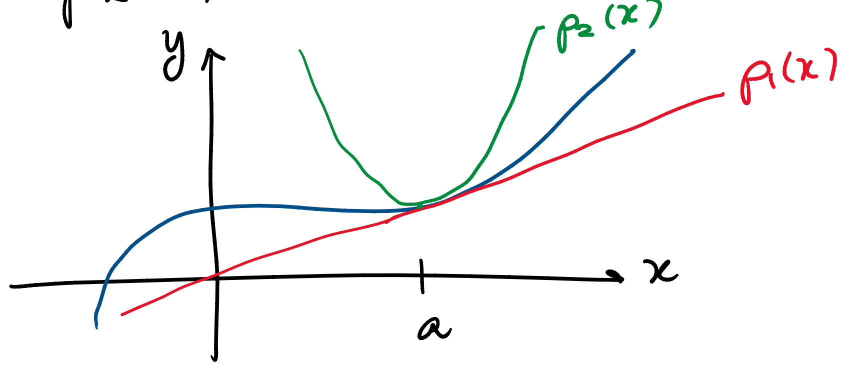
Let's impose $p''_2(a) = f''(a)$

$$p_2''(a) = \left[0 + 2c_2(1)\right]\Big|_{x=a} = 2c_2 = f''(a)$$

$$\boxed{c_2 = \frac{f''(a)}{2}}$$

Quadratic Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$



$p_2(x)$ is a good approximation for more values of x

Ex (1): Find the linear and quadratic approx. of $f(x) = \ln(x)$ at $a = 1$

Ex (1): Find the ... at $a = 1$ of $f(x) = \ln(x)$

Lin App: $p_1(x) = f(1) + f'(1)(x-1)$

$$f(1) = \ln(1) = 0$$

$$f'(1) = \left[\frac{1}{x}\right]\Big|_{x=1} = 1$$

$$p_1(x) = 0 + (1)(x-1) = \boxed{x-1}$$

Quad App: $p_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$

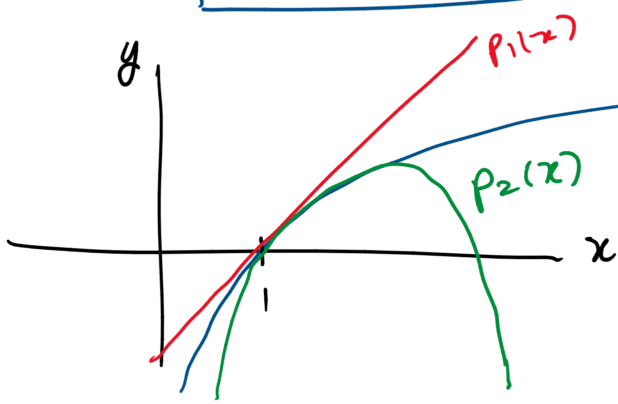
$$= 0 + (1)(x-1) + c_2(x-1)^2$$

Q: What is c_2?

$$c_2 = \frac{1}{2}f''(1) = \frac{1}{2}\left[\frac{-1}{x^2}\right]\Big|_{x=1} = \frac{-1}{2}$$

$$\boxed{p_2(x) = x-1 - \frac{1}{2}(x-1)^2}$$

Estimate $\ln(1.05)$



$$\ln(1.05) \approx p_1(1.05) = (x-1)\Big|_{1.05} = 0.05$$

$$\ln(1.05) \approx p_2(1.05) = (1.05-1) - \frac{1}{2}(1.05-1)^2$$

$$\ln(1.05) \approx p_2(1.05) = (1.05-1) - \frac{1}{2}(1.05-1)^2$$
$$= 0.05 - \frac{1}{2}(0.05)^2$$
$$= \boxed{0.04875}$$

Quad Approx is better

calculator $\ln(1.05) = 0.04879$

III. Taylor Polynomials :

Def: The n^{th}-order Taylor polynomial for $f(x)$ with center at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or we can write

$$p_n(x) = \sum_{k=0}^{n} c_k (x-a)^k \quad \text{where}$$

the coefficients $c_k = \dfrac{f^{(k)}(a)}{k!}$

k^{th} derivative of $f(x)$

Ex(2): Find $p_3(x)$ for $f(x) = \sin(x)$ centered at $a = 0$.

$$p_3(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3$$

$f(0) = 0$ $\qquad f^{(4)}(0) = 0$

$f'(0) = 1$ $\qquad f^{(5)}(0) = 1$

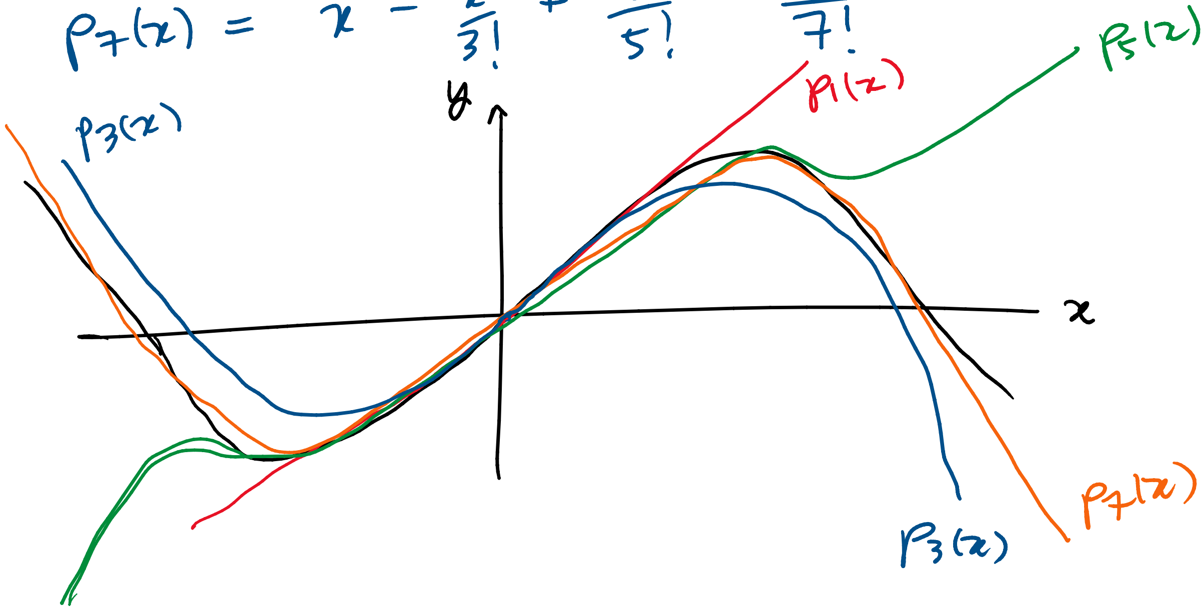
$f'(0) = 1$ $\qquad$ $f^{(5)}(0) = 1$

$f''(0) = 0$ $\qquad$ $f^{(6)}(0) = 0$

$f'''(0) = -1$ $\qquad$ $f^{(7)}(0) = -1$ $\quad \ldots$

$$P_3(x) = x - \frac{1}{3!}x^3 = x - \frac{x^3}{6}$$

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



NOTE: Taylor Approximations are used in many fields (Eng., Physics, etc) to approximate functions

Ex (3): Use a 3rd order Taylor polynomial to approximate $e^{0.1}$

$f(x) = e^x$ $\qquad$ center $a = 0$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3$$

$$f(0) = e^0 = 1$$
$$f'(0) = \left[e^x\right|_{x=0} = 1$$
$$f''(0) = \left[e^x\right]\Big|_{x=0} = 1$$
$$f'''(0) = \left[e^x\right]\Big|_{x=0} = 1$$

$$P_3(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3$$

$$e^{0.1} \approx P_3(0.1) = 1 + 0.1 + \tfrac{1}{2}(0.1)^2 + \tfrac{1}{6}(0.1)^3$$
$$\boxed{P_3(0.1) = 1.105167}$$

calculator $e^{0.1} = 1.1051709$

<u>Absolute error</u> $\left|e^{0.1} - P_3(0.1)\right| = \left|1.105171 - 1.105167\right|$
$$= 4.3 \times 10^{-6}$$

n	$P_n(0.1)$	$\left\|e^{0.1} - P_n(0.1)\right\|$
0	1	1.1×10^{-1}
1	1.1	5.2×10^{-3}
2	1.105	1.7×10^{-4}
3	1.105167	4.3×10^{-6}