Lesson 27 - MA 166 LEC 286

Announcements:
Exam 2 Answer Key posted on Brightspace

11.1: Approximating Functions with Polynomials - Part 1

★ Warm Up: Determine if the following series converges

$$\sum_{k=1}^{\infty} \frac{\cot^{-1}\left(\frac{1}{k}\right)}{k^2}$$

$\boxed{\text{(a) converges}}$ (b) diverges

Comparison $\rightarrow$ $b_k = \frac{1}{k^2}$ p-series $p=2$ converges

L.C.T

Direct: $a_k \geq \frac{\frac{\pi}{2}}{k^2}$ $\left(\frac{\pi}{2} \sum \frac{1}{k^2}\right)$

L.C.T $L = \lim_{k\to\infty} \frac{\frac{\cot^{-1}\left(\frac{1}{k}\right)}{\cancel{\left(\frac{1}{k^2}\right)}}}{\cancel{\left(\frac{1}{k^2}\right)}} = \lim_{k\to\infty} \cot^{-1}\left(\frac{1}{k}\right) = \frac{\pi}{2}$

I. Power Series:

So far, we've been studying $\sum_{k=1}^{\infty} a_k$ where a_k are real numbers

Q: What if $a_k = a_k(x)$ a function of x

Def: A power series is an infinite series

Def: A <u>power</u> <u>series</u> is an infinite series of the form:

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

or, more generally

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots$$

where a – <u>center</u> of the series

c_k – <u>coefficients</u> are constant real numbers

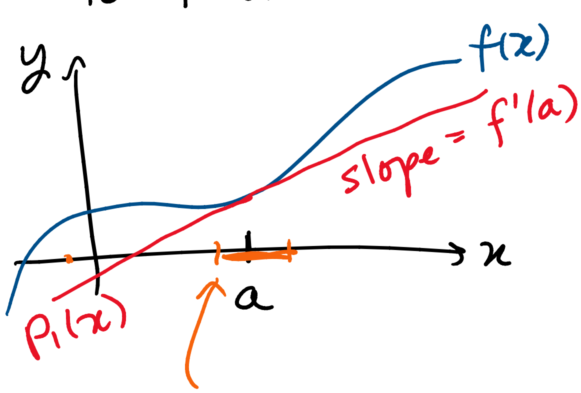
<u>GOAL</u>: Use power series to approximate a function $f(x)$

I. Linear Approximations:

In Calc 1, you found the <u>tangent</u> <u>line</u> to $f(x)$ at a

line

$$p_1(x) = f(a) + f'(a)(x-a)$$



We call $p_1(x)$ a <u>linear approximation</u> because its a line

When $|x-a|$ is small, then $p_1(x) \approx f(x)$ is a good approximation

To make the linear approximation better, let's add another term:

$$p_2(x) = \underbrace{f(a) + f'(a)(x-a)}_{p_1(x)} + \underbrace{c_2(x-a)^2}_{\text{quadratic term}}$$

Q: What is c_2?

$$p_2(a) = f(a) + f'(a)\cancel{(a-a)}^{\,0} + c_2\cancel{(a-a)}^{\,0\;2}$$

$$\boxed{p_2(a) = f(a)}$$

$$p_2'(a) = \left[0 + f'(a)(1) + 2c_2(x-a)\right]\Big|_{x=a}$$

$$= f'(a) + 2c_2\cancel{(a-a)}^{\,0}$$

$$\boxed{p_2'(a) = f'(a)} \rightarrow \text{have the same slope}$$

Let's impose that $p_2''(a) = f''(a)$

→ have the same curvature

$$f''(a) = p_2''(a) = \left[0 + 2c_2(1)\right]\Big|_{x=a} = 2c_2$$

$$\boxed{c_2 = \frac{1}{2} f''(a)}$$

Quadratic Approximation :

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

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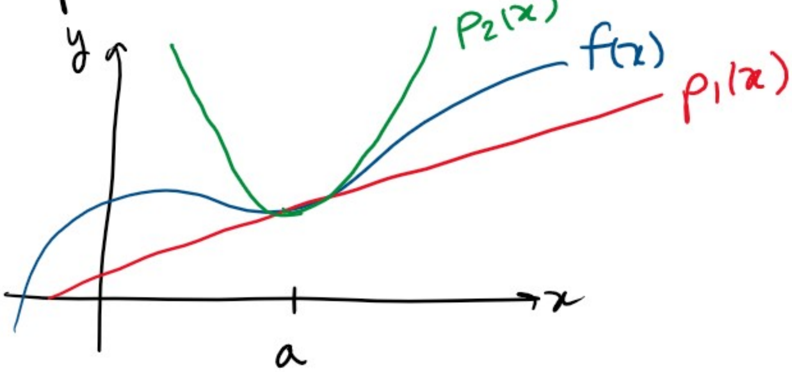


Claim:

the quadratic approx is better than the linear approx for $|x-a|$ small

Ex(1): Find the linear and quadratic approx of $f(x) = \ln(x)$ at $a=1$

Lin Approx: $p_1(x) = f(1) + f'(1)(x-1)$

$$f(1) = \ln(1) = 0$$

$$f'(1) = \left[\frac{1}{x}\right]\Big|_{x=1} = 1$$

$$p_1(x) = 0 + (1)(x-1) = \boxed{x-1}$$

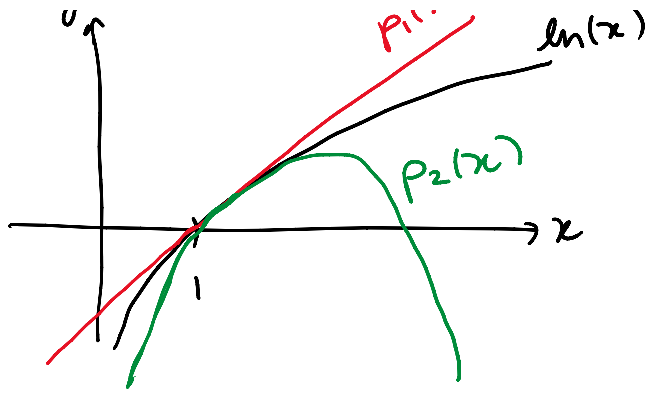
Quad Approx: $p_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$

$$= 0 + (1)(x-1) + c_2(x-1)^2$$

Find $c_2 = \frac{1}{2}f''(1) = \frac{1}{2}\left[\frac{-1}{x^2}\right]\Big|_{x=1} = \frac{-1}{2}$

$$p_2(x) = x - 1 - \frac{1}{2}(x-1)^2$$

claim:



claim:
$p_2(x)$ is better than $p_1(x)$ for x close to 1

Q: Use the approx.s to estimate $\ln(1.05)$

lin: $p_1(1.05) = (x-1)\big|_{1.05} = 1.05 - 1 = \boxed{0.05}$

quad: $p_2(1.05) = \left[x - 1 - \frac{1}{2}(x-1)^2\right]\Big|_{x=1.05}$

$= 0.05 - \frac{1}{2}(0.05)^2$

$= \boxed{0.04875}$

calculator: $\ln(1.05) = 0.04879$

quad approx is better

III. Taylor Polynomials :

Def: The nth-order Taylor Polynomial for $f(x)$ with center at a is:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or we can write

$$p_n(x) = \sum_{k=1}^{n} c_k (x-a)^k \quad \text{where}$$

the coefficients $c_k = \dfrac{f^{(k)}(a)}{k!}$ k^{th} derivative of $f(x)$

Ex (2): Find $p_3(x)$ for $f(x) = \sin(x)$ centered at $a=0$

$$p_3(x) = f(0) + f'(0)x + \tfrac{1}{2} f''(0)x^2 + \tfrac{1}{3!} f'''(0)x^3$$

$f(0) = 0$ $\qquad$ $f^{(4)}(0) = 0$

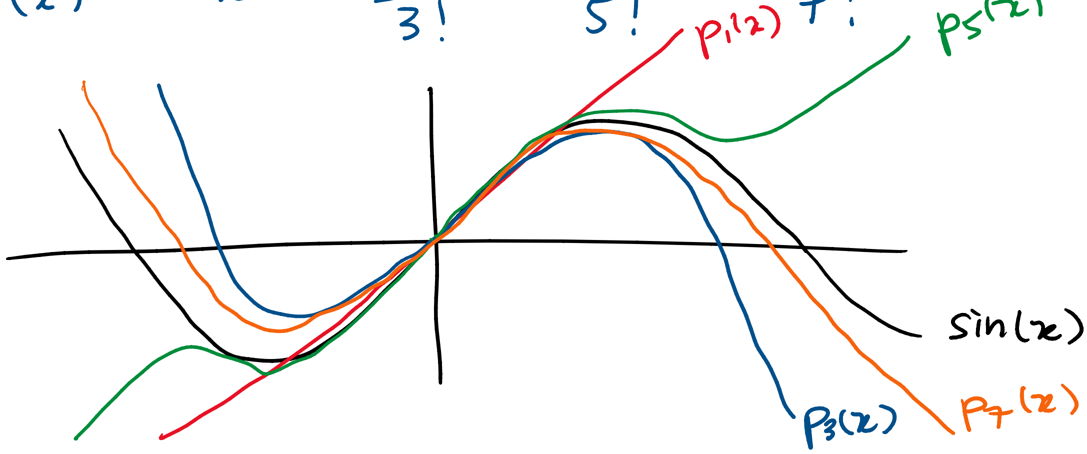
$f'(0) = 1$ $\qquad$ $f^{(5)}(0) = 1$

$f''(0) = 0$ $\qquad$ $f^{(6)}(0) = 0$

$f'''(0) = -1$ $\qquad$ $f^{(7)}(0) = -1$

$$p_3(x) = x - \frac{x^3}{3!}$$

$$p_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



NOTE: Taylor Approximations are a powerful tool in Science + Engineering tool in Science + Engineering

e.g., used to design computer algorithms to solve differential equations (Numerical Analysis)

Ex (3): Use a 3rd-order Taylor polynomial to approx $e^{0.1}$

$f(x) = e^x$ $\qquad a = 0$ ← closest whole number to 0.1

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3$$

$$f(0) = e^0 = 1$$

$$f'(0) = \left[e^x\right]\Big|_{x=0} = 1 \quad \cdots \quad f^{(k)}(0) = 1$$

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$e^{0.1} \approx P_3(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6}$$

$$= \boxed{1.105167}$$

calculator $e^{0.1} = 1.105171$

Absolute error $= \left| e^{0.1} - P_3(0.1) \right| = 4.3 \times 10^{-6}$

Q: What if we don't know $f(x)$?
How can we estimate $|f(x) - p_n(x)|$?

$\Rightarrow$ Next Class