11.1: Approximating with Polynomials - Part 2

☆ Warm Up:

Let $f(x) = (1+x)^{-3}$ and $a=0$. Find the coefficient c_4 in the 4th order Taylor polynomial of $f(x)$ centered at $a=0$

(a) $c_4 = -3$ (b) $c_4 = -10$ (c) $c_4 = 6$ $\boxed{\text{(d) } c_4 = 15}$

$$p_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$$

$$c_4 = \frac{f^{(4)}(0)}{4!} = \frac{\left[3\cdot4\cdot5\cdot6\,(1+x)^{-7}\right]\Big|_{x=0}}{4!}$$

$$= \frac{\cancel{3\cdot4}\cdot5\cdot6}{1\cdot2\cdot\cancel{3\cdot4}} = 5\cdot3 = \boxed{15}$$

I. Taylor Polynomials:

nth order Taylor polynomial of $f(x)$ centered at a

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

☆ Some calculators use Taylor polynomials to compute $\ln(x)$, $\sin(x)$, $\cos(x)$

Ex: $f(x) = \sqrt{x}$ Find $p_3(x)$ centered at $a=16$

$$p_3(x) = f(16) + f'(16)(x-16) + \frac{f''(16)}{2}(x-16)^2$$

Ex: $f(x) = \sqrt{x}$

$$P_3(x) = f(16) + f'(16)(x-16) + \frac{f''(16)}{2}(x-16)^2 + \frac{f'''(16)}{3!}(x-16)^3$$

$$f(16) = \sqrt{16} = 4$$

$$f'(16) = \frac{1}{2} x^{-1/2} \Big|_{x=16} = \frac{1}{2 \cdot 4} = \frac{1}{8}$$

$$f''(16) = \frac{1}{2} \cdot \frac{-1}{2} x^{-3/2} \Big|_{x=16} = \frac{-1}{4} \cdot (16)^{-3/2} = \frac{-1}{4} \cdot 4^{-3}$$

$$= \frac{-1}{4^4} = \frac{-1}{256}$$

$$f'''(16) = \frac{-1}{4} \cdot \frac{-3}{2} x^{-5/2} \Big|_{x=16} = \frac{3}{8} \cdot 4^{-5} = \frac{3}{8192}$$

$$P_3(x) = 4 + \frac{1}{8}(x-16) + \frac{\left(\frac{-1}{256}\right)}{2}(x-16)^2 + \frac{\left(\frac{3}{8192}\right)}{3!}(x-16)^3$$

$$\boxed{= 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2 + \frac{1}{16384}(x-16)^3}$$

NOTE: Use this to estimate $\sqrt{18}$

$$P_3(18) = 4 + \frac{1}{8}(2) - \frac{1}{512}(2)^2 + \frac{1}{16384}(2)^3$$

$$\boxed{P_3(18) = 4.242676} \approx \sqrt{18} = 4.242641$$

error: $|\sqrt{18} - P_3(18)| = |4.242676 - 4.242641|$

$$= \boxed{3.5 \times 10^{-5}}$$

Q: What if we don't know how to evaluate $f(x)$? How can we estimate the error?

II. Remainder of a Taylor polynomial:

$$R_n(x) = f(x) - p_n(x)$$

Just like with series

assume this converges to $f(x)$

Taylor series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x)$

Then $\displaystyle p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ partial sum

$$R_n(x) = f(x) - p_n(x) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the remainder

Taylor's (Remainder) Theorem:

Let $p_n(x)$ be the nth order Taylor polynomial of $f(x)$ centered at a

Then $\displaystyle R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some point c between x and a.

Idea: case $n=0$ $\quad p_n(x) = f(a)$

Mean Value Theorem says

$$\frac{f(x) - f(a)}{x-a} = f'(c)$$

for some c between x and a

Rearrange : $f(x) = \underbrace{f(a)}_{P_0(x)} + \underbrace{f'(c)(x-a)}_{R_0(x)}$

a similar arguments works for larger n.

Theorem : Estimate the Remainder :

Suppose $|f^{(n+1)}(c)| \leq M$

for all c between x and a

Then $|R_n(x)| \leq \dfrac{M|x-a|^{n+1}}{(n+1)!}$

Ex(2) : Estimate the remainder for $\sqrt{18}$ using

$$P_3(x) = 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2 + \frac{1}{16384}(x-16)^3$$

Want : $|R_3(18)| \leq M \dfrac{|18-16|^4}{4!} = \dfrac{M\,2^4}{4!} = \dfrac{M16}{24} = \dfrac{2}{3}M$

Need to find M

$$M = \max_{16 \leq c \leq 18} f^{(4)}(c) = \max_{16 \leq c \leq 18} -\frac{15}{16} c^{-7/2}$$

$$= \max_{16 \leq c \leq 18} \frac{-15}{16\, c^{7/2}}$$ decreasing in c max is at $c=16$

$$= \frac{-15}{16\,(16)^{7/2}} = \frac{-15}{16 \cdot 4^7} \approx 5.7 \times 10^{-5}$$

So $\boxed{|R_3(18)| \leq \frac{2}{3} M \approx 3.8 \times 10^{-5}}$

... 3rd order

Ex (3): Approximate $\ln\left(\frac{1}{2}\right)$ using the 3rd order Taylor polynomial for $f(x) = \ln(1+x)$ centered at $a = 0$

→ Estimate the error $\left| f\left(\tfrac{1}{2}\right) - P_3\left(\tfrac{1}{2}\right) \right|$

$$f(x) = \ln(1+x)$$
$$f'(x) = \frac{1}{1+x}$$
$$f''(x) = \frac{-1}{(1+x)^2}$$
$$f'''(x) = \frac{-2}{(1+x)^3}$$
$$f'''(0) = -2$$

$$P_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

Estimate $R_3(x)$ at $x = \frac{-1}{2}$

$$\left|R_3\left(\tfrac{-1}{2}\right)\right| \le \frac{M\,|x-a|^{3+1}}{(3+1)!}$$
$$= \frac{M\left|-\frac{1}{2} - 0\right|^4}{4!} = \frac{M}{2^4 \cdot 4!}$$

$$M = \max_{-\frac{1}{2} \le c \le 0} \left| f^{(4)}(c) \right| = \max_{-\frac{1}{2} \le c \le 0} \left| \frac{-6}{(1+c)^4} \right|$$

sign error in class

largest when $c = -\frac{1}{2}$

$$= \left| \frac{-6}{\left(1 + \frac{-1}{2}\right)^4} \right| = \frac{6}{\left(\frac{1}{2}\right)^4} = \frac{2^4 \cdot 6}{\cancel{3^4}} = 6 \cdot 2^4$$

$$\cancel{\left|R_3\left(-\tfrac{1}{2}\right)\right| \le \left(\frac{2^4 \cdot 6}{3^4}\right) \cdot \frac{1}{2^4 \cdot 4!} = \frac{6}{3^4 \cdot 4!} = \frac{2}{3^3 \cdot 24} = \frac{1}{27 \cdot 12} =}$$

$$= \frac{1}{4}$$

$$\boxed{\left|R_3\left(\frac{-1}{2}\right)\right| \le \frac{6 \cdot \cancel{2^4}}{\cancel{2^4} \cdot 4!} = \frac{6}{24} = \frac{1}{4}}$$