11.1: Approximating with Polynomials - Part 2

Warm Up:

Let $f(x) = (1+x)^{-3}$ and $a=0$. Find the coefficient c_4 in the 4th order Taylor polynomial of $f(x)$ centered at $a=0$

(a) $c_4 = -3$ (b) $c_4 = -10$ (c) $c_4 = 6$ $\boxed{\text{(d) } c_4 = 15}$

$$p_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$$

$$f'(x) = -3(1+x)^{-4}$$
$$f''(x) = 3\cdot 4(1+x)^{-5}$$
$$\vdots$$
$$f^{(4)}(x) = 3\cdot 4\cdot 5\cdot 6(1+x)^{-7}$$

$$c_4 = \frac{f^{(4)}(0)}{4!}$$

$$= \frac{\cancel{3}\cdot\cancel{4}\cdot 5\cdot 6}{1\cdot 2\cdot \cancel{3}\cdot\cancel{4}}$$

$$= 5\cdot 3 = \boxed{15}$$

I. Taylor Polynomials:

nth order Taylor polynomial of $f(x)$ centered at a

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

★ Some calculators use Taylor polynomials to calculate $\ln(x)$, $\sin(x)$, $\cos(x)$

Ex(1): $f(x) = \sqrt{x}$ find $p_3(x)$ centered at $a=16$

... to estimate $\sqrt{18}$

Ex(1): $f(x)=\sqrt{x}$ find $p_3(x)$ centered at ...
Use it to estimate $\sqrt{18}$

$$p_3(x)=f(16)+f'(16)(x-16)+\frac{f''(16)}{2}(x-16)^2+\frac{f'''(16)}{3!}(x-16)^3$$

$f(16)=4$

$$f'(16)=\left(\tfrac{1}{2}x^{-1/2}\right)\Big|_{x=16}=\tfrac{1}{2}\cdot 16^{-1/2}=\frac{1}{2\cdot 4}=\frac{1}{8}$$

$$f''(16)=\left(\tfrac{1}{2}\cdot\tfrac{-1}{2}x^{-3/2}\right)\Big|_{x=16}=-\frac{1}{4}\cdot\frac{1}{4^3}=\frac{-1}{4^4}=\frac{-1}{256}$$

$$f'''(16)=\left(-\frac{1}{4}\cdot\frac{-3}{2}\cdot x^{-5/2}\right)\Big|_{x=16}=\frac{3}{8}\cdot\frac{1}{4^5}=\frac{3}{8192}$$

$$\boxed{p_3(x)=4+\frac{1}{8}(x-16)-\frac{1}{512}(x-16)^2+\frac{1}{16384}(x-16)^3}$$

estimate $\sqrt{18}\approx p_3(18)=4+\frac{1}{8}(2)-\frac{1}{512}(2^2)+\frac{1}{16384}\cdot(2^3)$

$$\approx\boxed{4.242676}$$

True value of $\sqrt{18}=4.242641$

Absolute error $\left|\sqrt{18}-p_3(18)\right|\approx\boxed{3.5\times 10^{-6}}$

Q: What if we don't know $f(x)$?
How can estimate the error $|f(x)-p_n(x)|$?

II. Remainder of a Taylor Polynomial

Similar to series $\quad R_n(x)=f(x)-p_n(x)$

Assume the series converges to $f(x)$

Taylor Series $\displaystyle\sum^{\infty} \frac{f^{(k)}(a)}{\ldots}(x-a)^k = f(x)$

converges to $f(x)$

<u>Taylor Series</u> $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x)$

Then $\displaystyle p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ ← partial sum

$$R_n(x) = f(x) - p_n(x) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

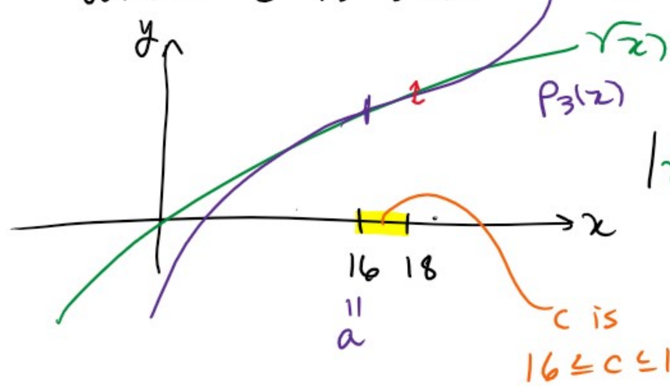
is the <u>remainder</u>

<u>Taylor's (Remainder) Theorem:</u>

Let $p_n(x)$ be the nth order Taylor polynomial of $f(x)$ centered at a

Then $$R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is between x and a



$$\left|\sqrt{18} - p_3(18)\right| = \frac{f^{(4)}(c)}{4!}(18-16)^4$$

<u>Idea:</u> case $n=0$, $p_0(x) = f(a)$

Mean Value Theorem

$$\frac{f(x)-f(a)}{x-a} = f'(c)$$ for some value c between x and a

rearrange

$$f(x) = f(a) + f'(c)(x-a)$$

Rearranging

$$f(x) = \underbrace{f(a)}_{P_0(x)} + \underbrace{f'(c)(x-a)}_{R_0(x)}$$

similar argument for larger n.

Theorem: Estimate the Remainder:

Suppose $|f^{(n+1)}(c)| \le M$ for all c between x and a

Then $$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}$$

Ex(2): Estimate $R_3(18)$ for $P_3(18) \approx \sqrt{18}$

$$|R_3(18)| \le M\frac{|x-a|^{3+1}}{(3+1)!} = M\frac{|18-16|^4}{4!}$$

$$= \frac{M \cdot 2^4}{4!} = \frac{M\cdot 16}{24} = \frac{2M}{3}$$

$$M = \max_{16 \le c \le 18} |f^{(4)}(c)| = \max_{16 \le c \le 18} \left| \frac{15}{16} c^{-7/2} \right|$$

$$= \max_{16 \le c \le 18} \left| \frac{-15}{16} \frac{1}{c^{7/2}} \right|$$

decreasing in c
max at $c = 16$

$$= \frac{15}{16}\frac{1}{16^{7/2}} = \frac{15}{16 \cdot 4^7} \approx 5.7 \times 10^{-5}$$

$$\boxed{|R_3(18)| \le \frac{2}{3}M \approx 3.8 \times 10^{-5}}$$

Absolute error $\quad |\sqrt{18} - P_3(18)| = 3.5 \times 10^{-5}$

Ex (3): We want to approx $\ln\left(\frac{1}{2}\right)$

Find the 3rd order T.P. of $\ln(1+x)$ centered at $x=0$

$$P_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(1-\frac{1}{2}\right) \approx P_3\left(\frac{-1}{2}\right) = -\frac{2}{3}$$

Q: Estimate the remainder $R_3(x)$.

$$\left|R_3\left(-\frac{1}{2}\right)\right| \le \frac{M|x-a|^{n+1}}{(n+1)!}$$

$$= \frac{M\left|\frac{-1}{2}-0\right|^4}{4!}$$

$$= \frac{M}{2^4 \cdot 4!}$$

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{2}{(1+x)^3}$

$f^{(4)}(x) = \frac{-6}{(1+x)^4}$

$a=0$

$x = -\frac{1}{2}$

Find M: $$M = \max_{-\frac{1}{2} \le c \le 0} \left|f^{(4)}(c)\right|$$

$$= \max_{-\frac{1}{2} \le c \le 0} \left|\frac{-6}{(1+c)^4}\right|$$ ← decreasing max is @ $c = -\frac{1}{2}$

$$= \left|\frac{-6}{\left(1-\frac{1}{2}\right)^4}\right| = \left|\frac{-6}{\frac{1}{2^4}}\right| = 6 \cdot 2^4$$

$$\left|R_3\left(\frac{-1}{2}\right)\right| \le \frac{M}{2^4 \cdot 4!} = \frac{6 \cdot \cancel{2^4}}{\cancel{2^4} \cdot 4!} = \frac{6}{24} = \frac{1}{4}$$

$$\boxed{\left|R_3\left(\frac{-1}{2}\right)\right| \le \frac{1}{4}}$$

$$\boxed{\left|R_3\left(-\tfrac{1}{2}\right)\right| \le \frac{1}{4}}$$