11.2 - Power Series - Part 1

Warm Up: Let $f(x) = \sin(x)$ and $a = 0$

The 2nd order Taylor polynomial of $f(x)$ centered at a is

$$p_2(x) = x$$

Estimate the error at $x = \frac{1}{2}$: $\left|f(\frac{1}{2}) - p_2(\frac{1}{2})\right|$

(a) $\frac{1}{60}$ $\quad$ $\boxed{\text{(b) } \frac{1}{48}}$ $\quad$ (c) $\frac{1}{24}$

$$\left|R_2\left(\tfrac{1}{2}\right)\right| \le \frac{M|x-a|^3}{3!} = \frac{M\left(\frac{1}{2}\right)^3}{6} = \frac{M}{8\cdot 6} = \frac{M}{48}$$

$$M = \max_{0 \le x \le \frac{1}{2}} |f'''(x)| = \max_{0 \le x \le \frac{1}{2}} |-\cos(x)| = 1$$

$$\left|R_2\left(\tfrac{1}{2}\right)\right| = \frac{1}{48}$$

Taylor Thm: $M = f'''(c)$ c is between a and x

I. Properties of Power Series:

power series: $\displaystyle\sum_{k=0}^{\infty} c_k (x-a)^k$

Ex: A Taylor Series of $f(x)$ is a power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Q: For what values of x does the Taylor series converge to $f(x)$?

Q: For what values of x does a power series converge?

Ex (1): $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}(x-2)^k$

For what values of x does it converge?

Ratio Test: $\displaystyle r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\cancel{(-1)^{k+1}}^{\,1}}{4^{\cancel{k}+1}}(x-2)^{k+1}\cdot\frac{\cancel{4^k}}{\cancel{(-1)^k}\,\cancel{(x-2)^k}}\right|$

$\displaystyle = \lim_{k\to\infty}\left|\frac{x-2}{4}\right| = \frac{|x-2|}{4}$

For convergence $r < 1$

$\displaystyle \frac{|x-2|}{4} < 1$ ← series converges

$|x-2| < 4$

$-4 < x-2 < 4$

$-2 < x < 6$

Recall, Ratio Test is inconclusive if $r=1$

Check: $x=-2$ and $x=6$

@ $x=6$: $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}(6-2)^k = \sum_{k=0}^{\infty}(-1)^k\frac{\cancel{4^k}}{\cancel{4^k}} = \sum_{k=0}^{\infty}(-1)^k$

A.S.T $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} 1 = 1$

→ series diverges

@ $x=-2$ $\quad \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}(-2-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k(-4)^k}{4^k}$

$= \sum_{k=0}^{\infty} (-1)^k(-1)^k = \sum_{k=0}^{\infty} (-1)^{2k} = \sum_{k=0}^{\infty} 1$

$\rightarrow$ series diverges

$\rightarrow$ series converges only if $-2 < x < 6$

<u>interval of convergence</u> $\quad I = (-2, 6)$

<u>radius of convergence</u> $\quad R = 4$

radius $R = \frac{6-(-2)}{2} = \frac{8}{2} = 4$

<u>NOTE</u>: The interval of convergence is always centered at a

<u>Ex (2)</u>: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\quad$ Q: Find the interval and radius of convergence.

Ratio Test $\quad r = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| = \lim_{k\to\infty} \left| \frac{x}{k+1} \right| = 0$

$r = 0 < 1 \rightarrow$ series converges for all x

$$\boxed{\begin{array}{l} I = (-\infty, \infty) \\ R = \infty \end{array}}$$

<u>Ex (3)</u>: $\sum_{k=0}^{\infty} k!\,x^k$

div by zero if $x=0$

if $x \neq 0$

Ex (3): $\sum_{k=0} k!\, x$

if $x = 0$ 1

Ratio Test $\quad r = \lim_{k\to\infty} \left| \frac{(k+1)!\, x^{k+1}}{k!\, x^k} \right| = \lim_{k\to\infty} \left| (k+1)x \right| \overset{*}{=} \infty$

$r = \infty > 1$ series diverges for all $x \neq 0$

if $x = 0$, $\sum_{k=0}^{\infty} k!\, 0^k = \sum_{k=0}^{\infty} 0 = 0$ converges to zero

$$\boxed{R = 0 \qquad I = [0, 0]}$$

Ex (4): The Taylor series for $f(x) = \ln(1+x)$ centered at $a = 0$ is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$$

start at $k=1$ b/c $f(0) = \ln(1) = 0$

Find the interval and radius of convergence.

Ratio Test: $r = \lim_{k\to\infty} \left| \frac{\cancel{(-1)^{k+2}}^{\,1}\, x^{\cancel{k+1}}}{k+1} \cdot \frac{k}{\cancel{(-1)^{k+1}}^{\,1}\, \cancel{x^k}} \right|$

$= \lim_{k\to\infty} \left| x \left(\frac{k}{k+1} \right) \right| = |x| \overbrace{\lim_{k\to\infty} \left(\frac{k}{k+1} \right)}^{=1}$

$r = |x| < 1$

series converges if $-1 < x < 1$

Check: $x=-1$ and $x=1$ → Does the Taylor series converge?

@ $x=-1$ $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k} = \sum_{k=1}^{\infty} \frac{-1}{k}$$

$$= -1\left(\sum_{k=1}^{\infty} \frac{1}{k}\right)$$ ← Harmonic series diverges

@ $x=1$ $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$ Alternating Harmonic Series

→ converges

T.S. converges $-1 < x \leq 1$

$$\boxed{\begin{array}{l} I = (-1, 1] \\ R = 1 \end{array}}$$

Ex(5): Find the Taylor Series of e^x centered at $a=0$ and find the interval and radius of convergence

$$e^x \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$f^{(k)}(0) = \left[\frac{d^k}{dx^k} e^x\right]\Bigg|_{x=0}$$

$$= \left[e^{x}\right]\Big|_{x=0} = 1$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \longleftarrow$$ Example 2 in lecture

$I = (-\infty, \infty)$

$R = \infty$

write " $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^{x}$ "