11.2 - Power Series - Part 1

☆ Warm Up: Let $f(x) = \sin(x)$ and $a = 0$

The 2nd order Taylor polynomial of $f(x)$ centered at a is

$$p_2(x) = x$$

Estimate the error at $x = \frac{1}{2}$: $\left|f(\tfrac{1}{2}) - p_2(\tfrac{1}{2})\right|$

(a) $\frac{1}{60}$ $\qquad$ $\boxed{\text{(b) } \frac{1}{48}}$ $\qquad$ (c) $\frac{1}{24}$

$$\left|R_2(\tfrac{1}{2})\right| \le \frac{M\,|x-a|^3}{3!} = \frac{M\,|\frac{1}{2}-0|^3}{3!} = \frac{M}{2^3\cdot 6} = \frac{M}{8\cdot 6} = \frac{M}{48}$$

$$M = \max_{a \le c \le x} |f'''(c)| = \max_{0 \le c \le \frac{1}{2}} |-\cos(c)| = 1$$

$$\boxed{\left|R_2(\tfrac{1}{2})\right| \le \frac{1}{48}}$$

I. Properties of Power Series

power series $\quad \displaystyle\sum_{k=0}^{\infty} c_k (x-a)^k$

Ex: The Taylor series of $f(x)$ is a power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Q: For what values of x does the Taylor Series converge?

Q: For what values of x does a power series converge?

Ex(1): $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}(x-2)^k$ For what x does the series converge?

Ratio Test: $\displaystyle r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\cancel{(-1)^{k+1}}^{\,1}(x-2)^{k+1}}{4^{k+1}}\cdot\frac{\cancel{4^k}}{\cancel{(-1)^k}_{\,1}\cancel{(x-2)^k}}\right|$

$\displaystyle = \lim_{k\to\infty}\left|\frac{x-2}{4}\right| = \frac{|x-2|}{4}$

converges if $r<1$

$\displaystyle \frac{|x-2|}{4} < 1$

$|x-2| < 4$

$-4 < x-2 < 4$

power series converges if $-2 < x < 6$

* Ratio Test is inconclusive when $r=1$

Need to check the endpoints

Check: $x=-2$, $x=6$

@ $x=-2$ $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}(-2-2)^k = \sum_{k=0}^{\infty}\frac{(-1)^k(-4)^k}{4^k}$

$\displaystyle\sum^{\infty}(-1)^k(-1)^k \quad \sum^{\infty}(-1)^{2k} \quad \sum^{\infty}$

$$= \sum_{k=0}^{\infty} (-1)^k (-1)^k = \sum_{k=0}^{\infty} (-1)^{2k} = \sum_{k=0}^{\infty} 1$$

$\rightarrow$ series diverges

@ $x = 6$ $$\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}(6-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k \cancel{4^k}}{\cancel{4^k}}$$

$$= \sum_{k=0}^{\infty} (-1)^k \rightarrow \text{diverges}$$

A.S.T $\lim_{k\to\infty} a_k = \lim_{k\to\infty} 1 = 1 \neq 0$

So the power series converges $-2 < x < 6$

<u>interval of convergence</u> $I = (-2, 6)$

<u>radius of convergence</u> $R = \frac{6-(-2)}{2} = \frac{8}{2} = 4$

<u>NOTE</u>: observe that the center of the interval of convergence is always a

$I \sim (a-R, a+R)$ ← inclusive or exclusive?

<u>Ex(2)</u>: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ Find the interval and the radius of convergence.

Ratio Test: $r = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| = \lim_{k\to\infty} \left| \frac{x}{k+1} \right| = 0$

$r = 0 < 1$ for all x

power series converges for all x

$$\boxed{\begin{array}{l} I = (-\infty, \infty) \\ R = \infty \end{array}}$$

Ex(3): $\displaystyle\sum_{k=0}^{\infty} k!\,x^k$

Ratio Test: $\displaystyle r = \lim_{k\to\infty} \left| \frac{(k+1)!\,x^{k+1}}{k!\,x^k} \right| = \lim_{k\to\infty} \left|(k+1)x\right| = +\infty$

$\displaystyle = |x| \lim_{k\to\infty} (k+1) = +\infty$

if $x=0$ this undefined

$r = +\infty > 1$ for $x \neq 0$

series diverges for all $x \neq 0$

@ $x=0$ $\displaystyle\sum_{k=0}^{\infty} k!\,0^k = \sum_{k=0}^{\infty} 0 = 0$ converges to zero

series converges at $x=0$

$$\boxed{R = 0 \qquad I = [0,0]}$$

Ex(4): The Taylor Series for $f(x) = \ln(1+x)$ centered at $x=0$ is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$$

Find the interval and radius of convergence

Find the interval and radius of convergence

Ratio Test: $r = \lim_{k\to\infty} \left| \frac{\cancel{(-1)^{k+2}}^{\,1}\, x^{\cancel{k}+1}}{k+1} \cdot \frac{k}{\underbrace{\cancel{(-1)^{k+1}}}_{1}\, \cancel{x^{k}}} \right|$

$$= \lim_{k\to\infty} \left| x \left(\frac{k}{k+1}\right)\right| = |x| \lim_{k\to\infty} \left(\frac{k}{k+1}\right) = |x|$$

$r = |x| < 1$

series converges for $-1 < x < 1$

<u>Check</u>: @ $x = -1$ and $x = 1$

@ $x = -1$ $\quad \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k}(-1)^k = \sum_{k=0}^{\infty} \frac{(-1)^{\overset{\text{odd}}{2k+1}}}{k} = \sum_{k=0}^{\infty} \frac{-1}{k}$

$$= -\left(\sum_{k=1}^{\infty} \frac{1}{k}\right)$$ ← Harmonic Series Diverges

series diverges @ $x = -1$

@ $x = 1$ $\quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ Alternating Harmonic Series

A.S.T $\quad a_k = \frac{1}{k}$ nonincreasing

$\lim_{k\to\infty} \frac{1}{k} = 0$ converges

so the Taylor series converges for $-1 < x \le 1$

$$\boxed{\begin{array}{l} I = (-1, 1] \\ R = 1 \end{array}}$$

" $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = \ln(1+x)$ for $-1 < x \leq 1$ "

Ex(5): Find the Taylor series of $e^x = f(x)$ centered at $a=0$ and find its interval and radius of convergence

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$f^{(k)}(0) = \left[\frac{d^k}{dx^k} e^x \right] \Big|_{x=0}$$

$$= \left(e^x \right] \Big|_{x=0} = e^0 = 1$$

T.S. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\leftarrow$ ex (2)

$I = (-\infty, \infty)$

$R = \infty$

" $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ for all x "