Lesson 2 - MA 166 - Spring 22

13.2 Vectors in 3D

13.5 Lines + planes in Space

Announcements:

- Lectures online Wed + Fri
- Office Hours by appointment only
- Recitations still in person

Warm up:

Find the unit vector $\vec{u}$ parallel to $\vec{v} = \langle 3, -2, 1 \rangle$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ $\qquad |\vec{v}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$

$$\boxed{\vec{u} = \frac{1}{\sqrt{14}} \langle 3, -2, 1 \rangle}$$

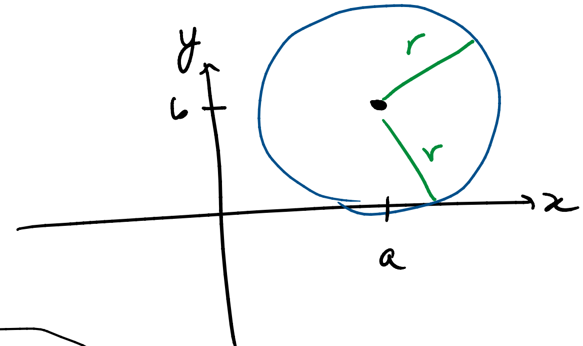
GOALS:

13.2 - Equation of a sphere

13.5 - Find equations of lines + planes

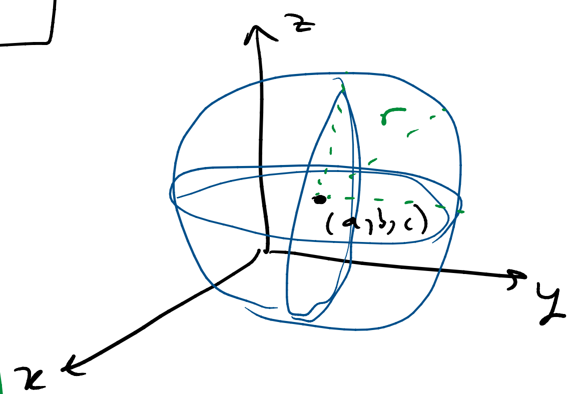
I. Spheres + circles:

In 2D a circle with center (a, b) radius r



$$\boxed{\begin{aligned} \sqrt{(x-a)^2 + (y-b)^2} &= r \\ (x-a)^2 + (y-b)^2 &= r^2 \end{aligned}}$$

In 3D, a sphere center: (a, b, c) radius $= r$



$$\boxed{\begin{aligned} \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} &= r \\ (x-a)^2 + (y-b)^2 + (z-c)^2 &= r^2 \end{aligned}}$$

Ex: The following equation represents a sphere

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$$x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$$

Find the center and radius of the sphere

1. collect like terms

$$x^2 - 2x + 1$$
$$+ y^2 + 6y + 3^2 \quad = -1 + 1 + 3^2 + 4^2$$
$$+ z^2 - 8z + 4^2$$

2. Complete the square (x, y, z)

$$x^2 - 2x + 1 = (x-1)^2$$
$$y^2 + 6y + 3^2 = (y+3)^2$$
$$z^2 - 8z + 4^2 = (z-4)^2$$

3. Simplify:

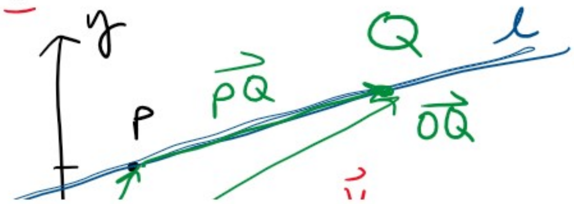
$$(x-1)^2 + (y+3)^2 + (z-4)^2 = \cancel{-1} + \cancel{1} + 9 + 16$$
$$= 25 = 5^2$$

$$\boxed{\begin{array}{ll} \text{center} & (1, -3, 4) \\ \text{radius} & r = 5 \end{array}}$$

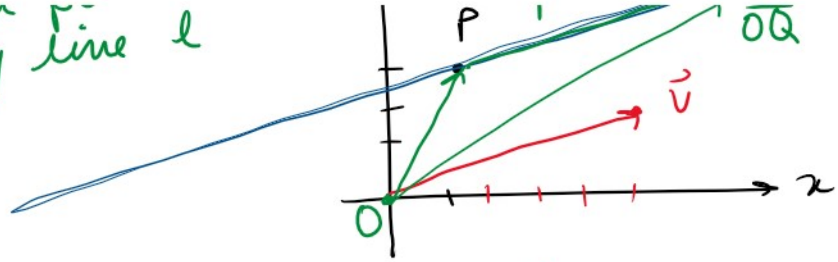
II. Equation of a line in 2D space

Let ℓ be the line
- passes through point $P = (1,3)$
- parallel to $\vec{v} = \langle 5, 2 \rangle$

Consider a point Q along line ℓ



Consider a p... Q along line ℓ



The location of Q is given by

$\vec{OQ} = \vec{OP} + \vec{PQ}$ vector addition

we know the line ℓ is $\parallel$ to $\vec{v}$

$\vec{PQ}$ is $\parallel$ to $\vec{v}$

$\vec{PQ} = t\vec{v}$ scalar multiple

t - unknown scalar

in terms of x and y

$$\boxed{\langle x, y \rangle = \langle 1, 3 \rangle + t\langle 5, 2 \rangle \quad \text{for } -\infty < t < \infty}$$

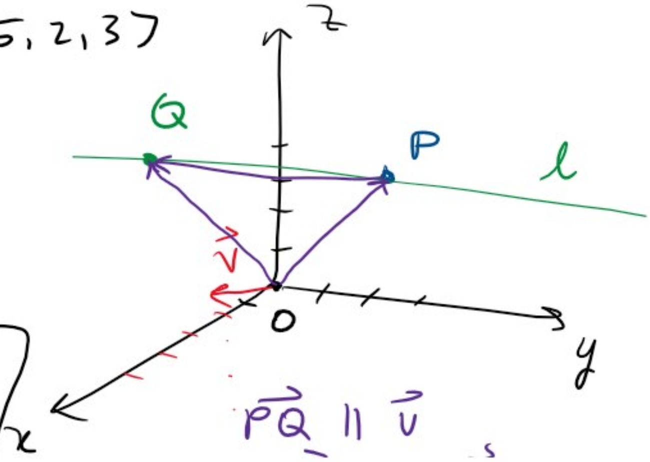
vector equation for line ℓ

To get the parametric equation of the line, separate out components of the vectors

$$\begin{cases} x = 1 + 5t & \text{for } -\infty < t < \infty \\ y = 3 + 2t \end{cases}$$

NOTE: This generalizes in 3D

Ex: Find a line through point $P = (1,3,4)$ parallel to $\vec{v} = \langle 5,2,3 \rangle$



$\vec{OQ} = \vec{OP} + \vec{PQ}$

$= \vec{OP} + t\vec{v}$

$\vec{PQ} \parallel \vec{v}$

$\langle x, y, z \rangle = \langle 1,3,4 \rangle + t\langle 5,2,3 \rangle$

$$\boxed{\langle x,y,z\rangle = \langle 1,3,4\rangle + t\langle 5,2,3\rangle \text{ for } -\infty < t < \infty}$$ x

$\vec{PQ} \parallel \vec{v}$

$\vec{PQ} = t\vec{v}$

POLL: (Not on Heat Seat)

Which of the following lines is parallel to the vector $\vec{v} = \langle 1,1,1\rangle$

(a) $\langle x,y,z\rangle = \langle 1,1,1\rangle + t\langle 1,2,3\rangle$

(b) $\boxed{\langle x,y,z\rangle = \langle 1,2,3\rangle + t\langle 1,1,1\rangle}$

(c) $\langle x,y,z\rangle = \langle 1,2,1\rangle + t\langle 3,2,1\rangle$

IV Equation of a Plane:

Def: A plane passing through point $P_0 = (x_0, y_0, z_0)$ with nonzero normal vector $\vec{n} = \langle a,b,c\rangle$ is described by

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

or

$$\boxed{ax + by + cz = d = ax_0 + by_0 + cz_0}$$

Ex: The equation of plane through $P_0 = (2,-3,4)$ with $\vec{n} = \langle -1,2,3\rangle$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$-1(x-2) + 2(y+3) + 3(z-4) = 0$$

pull constants over to RHS

$$-x + 2y + 3z = -2 - 6 + 12 = 4$$

$$\boxed{-x + 2y + 3z = 4}$$

<u>Def</u>: Given a fixed point P_0 and a nonzero normal vector $\vec{n}$, the set of points P in 3D space for which $\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$ ("perpendicular") is called a <u>plane</u>.

<u>Ex</u>: Find the equation of the plane passing through $P_0 = (2,-3,4)$ that is perpendicular to the line

$$\text{line} \quad \begin{cases} x = 3 + 2t \\ y = 0 - 4t \\ z = 1 - 6t \end{cases}$$

<u>GOAL</u>: Find our normal vector $\vec{n}$

First write ℓ in vector form

$$\langle x, y, z\rangle = \langle 3, 0, 1\rangle + t\langle 2, -4, -6\rangle$$

$\vec{v} = \langle 2,-4,-6\rangle$ is <u>parallel</u> to line ℓ

<u>WANT</u>: plane to be perpendicular to ℓ
normal vector $\vec{n}$ is perpendicular to plane

Take $\quad \vec{v} = \vec{n} = \langle 2,-4,-6\rangle \quad P_0 = (2,-3,4)$

Plane $\quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$$2(x-2) - 4(y+3) - 6(z-4) = 0$$

$$2x - 4y - 6z = 4 + 12 - 24 = -8$$

(divide both sides by 2)

$$\boxed{x - 2y - 3z = -4}$$

<u>Def</u>: Two planes are <u>parallel</u> if their respective normal vectors are parallel

vectors are scalar multiples

normal vectors are parallel
(i.e. the normal vectors are scalar multiples of each other)

Ex: Find the equation of the plane Q that passes through point $(-2,4,1)$ and is parallel to the plane R:

$$3x - 2y + z = 4$$

1. Find the normal vector to R
$$\vec{n} = \langle 3, -2, 1 \rangle$$

2. Q: point $(-2,4,1)$
$$\vec{n} = k\langle 3, -2, 1 \rangle$$ where k is a scalar

Take $k=1$

$$3(x+2) + (-2)(y-4) + (1)(z-1) = 0$$

$$3x - 2y + z = -6 - 8 + 1 = -13$$

$$\boxed{3x - 2y + z = -13}$$

POLL: Which of the following planes is parallel to the plane Q

$$2x - 3y + 5z = 7 \qquad \vec{n} = \langle 2, -3, 5 \rangle$$

(a) $5x - 3y + 2z = 1$ $\qquad \vec{n} = \langle 5, -3, 2 \rangle$

(b) $7x + 2y - 3z = 5$ $\qquad \vec{n} = \langle 7, 2, -3 \rangle$

★ (c) $4x - 6y + 10z = 1$ $\qquad \vec{n} = \langle 4, -6, 10 \rangle$

(d) $2x + 3y + 5z = -8$ $\qquad \vec{n} = \langle 2, 3, 5 \rangle$

Which $\vec{n}$ is a scalar multiple of $\langle 2, -3, 5 \rangle$?

which $\vec{n}$ is a scalar multiple of $\langle 2,-3,5\rangle$?

☆ Summary:

- equation of a sphere w/ radius r and center (a,b,c)

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

- equation of a line through $P_0 = (x_0, y_0, z_0)$ and parallel to vector $\vec{v} = \langle v_1, v_2, v_3\rangle$

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle v_1, v_2, v_3\rangle$$

- equation of a plane through $P_0 = (x_0, y_0, z_0)$ w/ normal vector $\vec{n} = \langle a, b, c\rangle$ is

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$