Lesson 30 - MA 166 LEC 200

Announcements:

Exam 3 is on Wed Apr 20 @ 6:30pm in ELLT

- study guide posted on BS

11.3 & 11.4:

Summary of Taylor Series and Applications

☆ **Warm Up:** Find the radius of convergence, R, of

$$\sum_{k=1}^{\infty} \frac{x^{2k+1}}{2^{k-1}}$$

$k \mapsto k+1$

(a) $R = 2$

(b) $R = \frac{1}{2}$

(c) $R = \infty$

(d) $\boxed{R = \sqrt{2}}$

(e) $R = 0$

Ratio Test

$$r = \lim_{k\to\infty} \left| \frac{x^{2(k+1)+1}}{2^{k+1-1}} \cdot \frac{2^{k-1}}{x^{2k+1}} \right| = \lim_{k\to\infty} \left| \frac{x^{2k+3}}{2^k} \cdot \frac{2^{k-1}}{x^{2k+1}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x^2}{2} \right| = \frac{|x|^2}{2} < 1$$

$$|x|^2 < 2$$

$$|x| < \sqrt{2} \leftarrow \text{Radius of conv.}$$

$$\boxed{R = \sqrt{2}}$$

I. Taylor & Maclaurin series:

Taylor series of $f(x)$ centered at a

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Taylor series of $f(x)$

$k=0 \quad k.$

Maclaurin Series is a Taylor series of $f(x)$ centered at $a=0$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Some common Maclaurin series

① $$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

② $$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

③ $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Ex.(1): (a) Find the Maclaurin series $f(x) = \frac{1}{1-x}$

(b) Find the interval of convergence

(a) $$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$f(0) = \frac{1}{1-0} = 1$

$f'(0) = \left[\frac{-1}{(1-x)^2} \cdot (-1) \right] \Big|_{x=0}$

$$= \frac{1}{(1-0)^2} = 1$$

$$f''(0) = \left[\frac{-2}{(1-x)^3} \cdot (-1) \right] \Big|_{x=0} = \frac{2}{(1-0)^3} = 2$$

$$f'''(0) = \left[\frac{2 \cdot (-3)}{(1-x)^4} \cdot (-1) \right] \Big|_{x=0} = \frac{3!}{(1-0)^4} = 3!$$

$$\vdots$$

$$f^{(k)}(0) = \frac{k!}{(1-x)^{k+1}} \Big|_{x=0} = k!$$

$$\sum_{k=0}^{\infty} \frac{k!}{k!} x^k = \sum_{k=0}^{\infty} x^k$$

(b) Find the interval of convergence

Ratio Test $\quad r = \lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k} \right| = \lim_{k \to \infty} |x| = |x| < 1$

$$\boxed{R = 1} \qquad -1 < x < 1$$

Check endpoints:

@ $x = 1$ $\quad \sum_{k=0}^{\infty} 1^k = \sum_{k=0}^{\infty} 1$ $\quad$ diverges

@ $x = -1$ $\quad \sum_{k=0}^{\infty} (-1)^k$ $\quad$ diverges

$$\lim_{k \to \infty} a_k = 1$$

$$\boxed{I = (-1, 1)}$$

NOTE: (How to remember on a test)

Notice $\sum_{k=0}^{\infty} x^k$ Geometric series converges $|x| < 1$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

II. Applications:

What is the Maclaurin series of

$$f(x) = \frac{x^5}{1-x}$$

we know $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

so $$\frac{x^5}{1-x} = x^5\left(\frac{1}{1-x}\right) = x^5\left(\sum_{k=0}^{\infty} x^k\right)$$

$$= \sum_{k=0}^{\infty} x^5 \cdot x^k = \sum_{k=0}^{\infty} x^{k+5} = x^5 + x^6 + x^7 + \dots$$

Rewrite: $\sum_{k=5}^{\infty} x^k$ ← Geometric series converges $|x| < 1$

$$\boxed{R = 1 \quad I = (-1, 1)}$$

Ex (2): Find the Maclaurin series of

$$f(x) = \frac{1}{1-2x}$$

and find interval of convergence.

$$f(x) = \frac{1}{1-2x}$$ and find interval of convergence.

Let $y = 2x$ $\qquad \frac{1}{1-y} = \sum_{k=0}^{\infty} y^k$

$$\frac{1}{1-2x} = \sum_{k=0}^{\infty} (2x)^k = \sum_{k=0}^{\infty} 2^k x^k$$

Converges if $|y| < 1$

$$|2x| < 1$$

$$|x| < \frac{1}{2}$$

$$\boxed{R = \frac{1}{2} \qquad I = \left(-\frac{1}{2}, \frac{1}{2}\right)}$$

Ex(3): Find the Maclaurin series of

$$f(x) = \frac{1}{1+x^2}$$

$$\frac{1}{1-y} = \sum_{k=0}^{\infty} y^k \qquad \text{Let } y = -x^2$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

converges: if $|y| < 1$

$$|-x^2| < 1$$

$|x^2| < 1$

$|x| < 1$ $\boxed{I = (-1, 1)}$

III. Limits:

Evaluate $\lim_{x \to 0} \dfrac{x^2 + 2\cos(x) - 2}{3x^4} = \dfrac{0}{0}$

Could use L'Hopital's $\longrightarrow$ apply it 4x

Instead use $\cos(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \cdots$

$$= \lim_{x \to 0} \frac{x^2 + 2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) - 2}{3x^4}$$

$$= \lim_{x \to 0} \frac{(\cancel{x^2} \cancel{-2}) + \left(\cancel{2} - \cancel{x^2} + \frac{x^4}{12} - \frac{x^6}{360} + \cdots\right)}{3x^4}$$

$$= \lim_{x \to 0} \frac{\frac{x^4}{12} - \frac{x^6}{360} + \cdots}{3x^4} = \lim_{x \to 0} \frac{1}{36} - \overbrace{\frac{x^2}{1080}}^{\to 0} + \overbrace{\cdots}^{\to 0}$$

all of these terms have x^k where $k \geq 4$

$$= \boxed{\frac{1}{36}}$$

Ex (4): Evaluate $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{x}$

TS $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

T.S. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= \lim_{x \to 0} \frac{\left(\cancel{1} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cdots\right) - \left(\cancel{1} - x + \cancel{\frac{x^2}{2!}} - \frac{x^3}{3!} + \cdots\right)}{x}$$

$$= \lim_{x \to 0} \frac{2x + \frac{2x^3}{3!} + \cdots}{x} = \boxed{2}$$