Lesson 30 - MA 166 LEC 286

11.3 & 11.4:

Summary of Taylor Series and Applications

Announcements:

Exam 3 is on Wed Apr 20 @ 6:30pm in ELLT
- study guide posted on BS

☆ Warm Up: Find the radius of convergence, R, of

$$\sum_{k=1}^{\infty} \frac{x^{2k+1}}{2^{k-1}}$$

$k \longmapsto k+1$

(a) $R = 2$

(b) $R = \frac{1}{2}$

(c) $R = \infty$

$\boxed{(d)\ R = \sqrt{2}}$

(e) $R = 0$

Ratio Test

$$r = \lim_{k\to\infty} \left| \frac{x^{2(k+1)+1}}{2^{(k+1)-1}} \cdot \frac{2^{k-1}}{x^{2k+1}} \right| = \lim_{k\to\infty} \left| \frac{x^{\cancel{2k+3}\,2}}{2^{k}} \cdot \frac{2^{\cancel{k-1}}}{x^{\cancel{2k+1}}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x^2}{2} \right| = \frac{|x|^2}{2} < 1$$

$$|x|^2 < 2$$

$$|x| < \sqrt{2}$$

$$\boxed{R = \sqrt{2}}$$

$$I = (-\sqrt{2}, \sqrt{2})$$

check endpoints:

$$\sum \frac{(-\sqrt{2})^{2k+1}}{2^{k-1}}$$

I. Taylor & Maclaurin Series:

Taylor Series of $f(x)$ centered at a

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Maclaurin Series is a Taylor series of $f(x)$ centered at $a = 0$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$\sum_{k=0} \frac{f^{(k)}(0)}{k!} x^k$$

Some common Maclaurin series

① $$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

② $$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

③ $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Ex(1): (a) Find the Maclaurin series $f(x) = \frac{1}{1-x}$

(b) Find the interval of convergence

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$f(0) = \frac{1}{1-0} = 1 = 0!$$

$$f'(0) = \left[\frac{-1}{(1-x)^2} \cdot (-1)\right]\Big|_{x=0} = \frac{1}{(1-0)^2} = 1 = 1!$$

$$f''(0) = \left[\frac{-2}{(1-x)^3} \cdot (-1)\right]\Big|_{x=0} = \frac{2}{(1-0)^3} = 2 = 2!$$

$$f'''(0) = \left[\frac{2\cdot(-3)}{(1-x)^4} \cdot (-1)\right]\Big|_{x=0} = \frac{3!}{(1-0)^4} = 3!$$

$$f^{(k)}(0) = \left[\frac{k!}{(1-x)^{k+1}}\right]\Big|_{x=0} = k!$$

$$f^{(k)}(0) = \left[\frac{k!}{(1-x)^{k+1}}\right]\Bigg|_{x=0} = k!$$

Maclaurin series: $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{k!} x^k = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

(b) Find the interval of convergence

Ratio Test: $\displaystyle r = \lim_{k\to\infty} \left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty} |x| = |x| < 1$

$$\boxed{R = 1}$$

Ratio Test is inconclusive if $r=1$ $\rightarrow$ check endpoints

@ $x = 1$ $\displaystyle\sum_{k=1}^{\infty} 1^k = \sum_{k=1}^{\infty} 1$ diverges

@ $x = -1$ $\displaystyle\sum_{k=1}^{\infty} (-1)^k$ diverges $\displaystyle\lim_{k\to\infty} 1 = 1$ A.S.T.

$$\boxed{I = (-1, 1)}$$

NOTE: How to Remember on a Test

Notice $\displaystyle\sum_{k=0}^{\infty} x^k$ is a Geometric series $|x| < 1$ converges

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \Rightarrow I = (-1, 1)$$

II. Applications:

What is the Maclaurin series of

$$f(x) = \frac{x^5}{1-x}$$

$$\frac{x^5}{1-x} = x^5\left(\frac{1}{1-x}\right) = x^5\left(\sum_{k=0}^{\infty} x^k\right)$$

$$\sum^{\infty} x^5 x^k = \sum^{\infty} x^{k+5}$$

$1-x$

$$= \sum_{k=0}^{\infty} x^5 \cdot x^k = \sum_{k=0}^{\infty} x^{k+5}$$

$$= x^5 + x^6 + x^7 + x^8 + \cdots$$

$$= \sum_{k=5}^{\infty} x^k$$ ← Geometric series converges if $|x|<1$

$$\boxed{I = (-1, 1)}$$

Ex(2): Find the Maclaurin series of

$$f(x) = \frac{1}{1-2x}$$

$$\frac{1}{1-y} = \sum_{k=0}^{\infty} y^k \qquad \text{let } y = 2x$$

$$\frac{1}{1-(2x)} = \sum_{k=0}^{\infty} (2x)^k = \sum_{k=0}^{\infty} 2^k x^k$$

Converges $|y| < 1$

$|2x| < 1$

$|x| < \frac{1}{2}$

$$\boxed{R = \frac{1}{2}, \ I = \left(-\frac{1}{2}, \frac{1}{2}\right)}$$

Ex(3): Find the Maclaurin series of

$$f(x) = \frac{2x^3}{1+x^2}$$

and find the interval of convergence.

$$f(x) = \frac{2x^3}{1+x^2} = 2x^3\left(\frac{1}{1-y}\right) = 2x^3 \sum_{k=0}^{\infty} y^k$$

let $y = -x^2$ $$= 2x^3 \sum_{k=0}^{\infty} (-x^2)^k$$

$$= 2x^3 \sum_{k=0}^{\infty} (-1)^k x^{2k} = 2 \sum_{k=0}^{\infty} (-1)^k x^{2k+3}$$

A.S.T.

$$\lim_{k\to\infty} x^{2k+3} = 0$$

$|x| < 1$

Then converges if $|y| < 1$

$|-x^2| < 1$

$|x^2| < 1$

$\boxed{I = (-1, 1)}$ $\quad |x| < 1$

III. Limits:

Evaluate $\lim_{x \to 0} \dfrac{x^2 + 2\cos(x) - 2}{3x^4} = \dfrac{0}{0}$

Could use L'Hopital's $\longrightarrow$ apply 4x

Instead use the Maclaurin series for $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

$$\lim_{x \to 0} \frac{x^2 + 2\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right] - 2}{3x^4}$$

$$= \lim_{x \to 0} \frac{(\cancel{x^2} \cancel{-2}) + \left[\cancel{2} - \cancel{x^2} + \frac{x^4}{12} - \frac{x^6}{360} + \cdots\right]}{3x^4}$$

$$= \lim_{x \to 0} \frac{\frac{x^4}{12} - \frac{x^6}{360} + \cdots}{3x^4} = \lim_{x \to 0} \frac{1}{36} - \frac{x^2}{1080} + \cdots$$

$\to 0$

all of these terms $c_k x^k$ where $k \geq 4$ $\longrightarrow 0$

$$= \boxed{\frac{1}{36}}$$

Ex(4): $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{x}$

use $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \cdots$

$e^{-x} = 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$$\lim_{x\to 0} \frac{\left(\cancel{1} + x + \cancel{\frac{x^2}{2}} + \frac{x^3}{6} + \cdots\right) - \left(\cancel{1} - x + \cancel{\frac{x^2}{2}} - \frac{x^3}{6} + \cdots\right)}{x}$$

$$= \lim_{x\to 0} \frac{2x + \frac{2x^3}{6} + \cdots}{x} = \boxed{2}$$